SSD and other energy deformations

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energy deformation

uniform system

$$\mathcal{H} = \sum_x \hat{h}_x$$

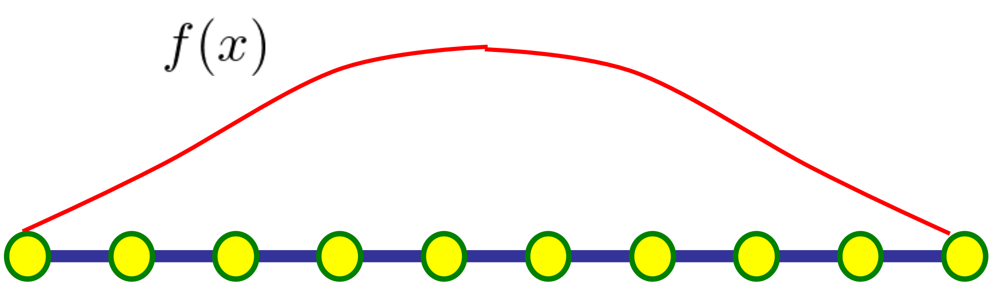
($\hat{h}_x$: local Hamiltonian)

energy-scale deformation

$$\mathcal{H}_{\rm def} = \sum_x f(x)\, \hat{h}_x$$

$f(x)$: deformation function

slowly varying



Outline

1. Energy deformation (before SSD)

    smooth boundary condition
    classical model in curved plane
        Hyperbolic deformation

2. Sine-Square Deformation

    spherical (sine) deformation
    periodic ground state in SSD system
    Application of SSD
        grand-canonical SSD
        long-distance entanglement in SD-α system

3. Discussions

    extend SSD physics
    explore useful properties of various energy deformation

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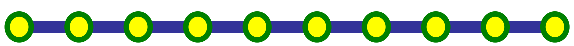
◆ Boundary effect

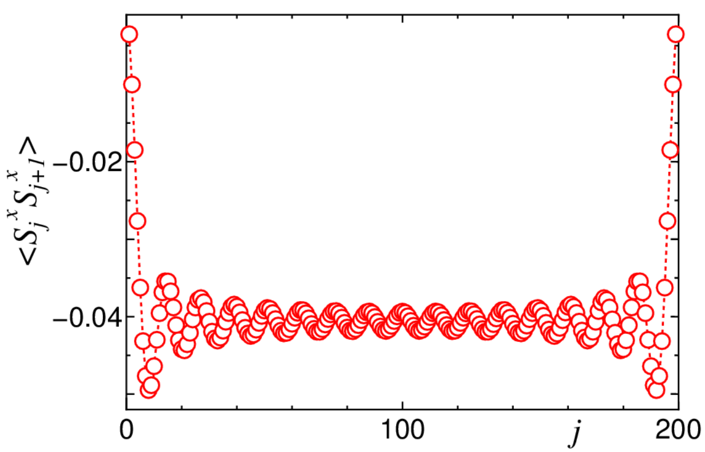
Open boundary induces

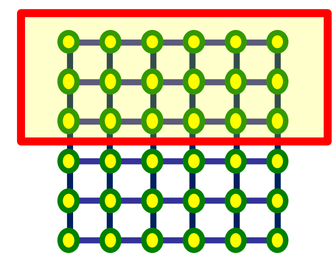
boundary oscillation (*e.g.* Friedel oscillation)

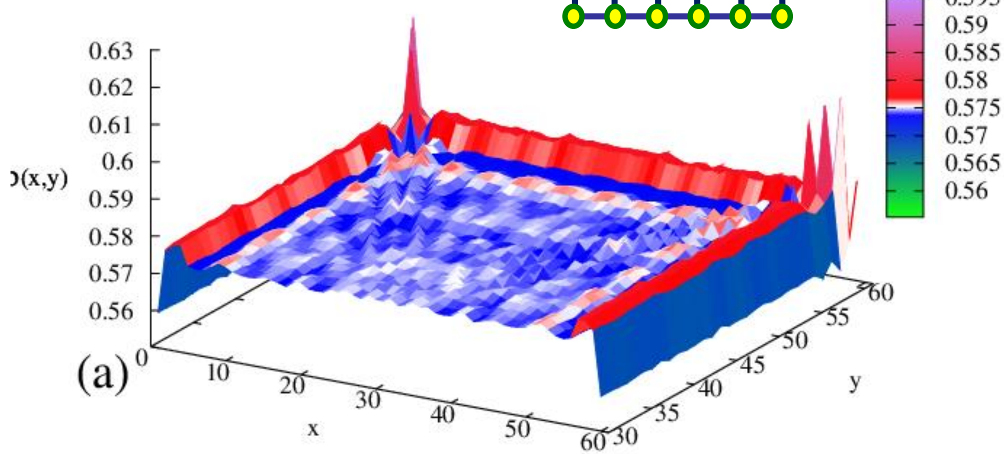
large size-dependence (*e.g.* energy/site ~ N^{-1})

(effective) degree of freedom at system edges









◆ Smooth boundary conditions

Boundary effect :

(sometimes) an obstacle for studying thermodynamic limit

treatment to remove/ lessen boundary effects is desirable

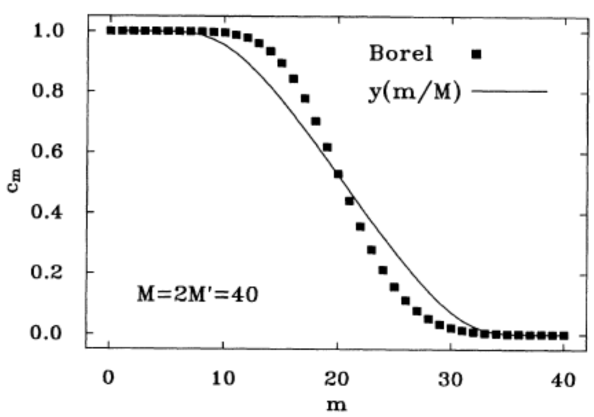
Smooth boundary condition

Vekic-White [PRL 71, 4283 (1993), PRB 53, 14552 (1996)]

"smoothly turn off (set to zero) the parameters of the Hamiltonian near the edges of the system"

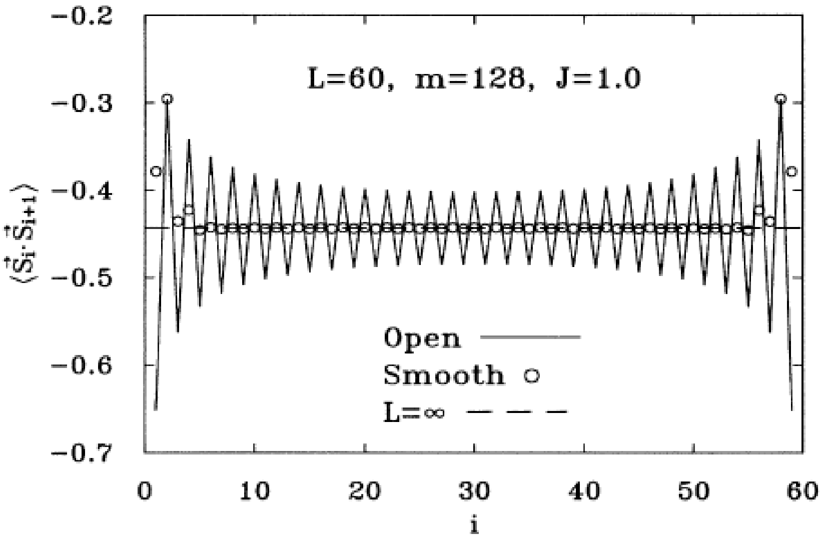
$$y(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x - 1/2}{x(1-x)}\right)\right]$$

around the edges of 1D/2D quantum systems

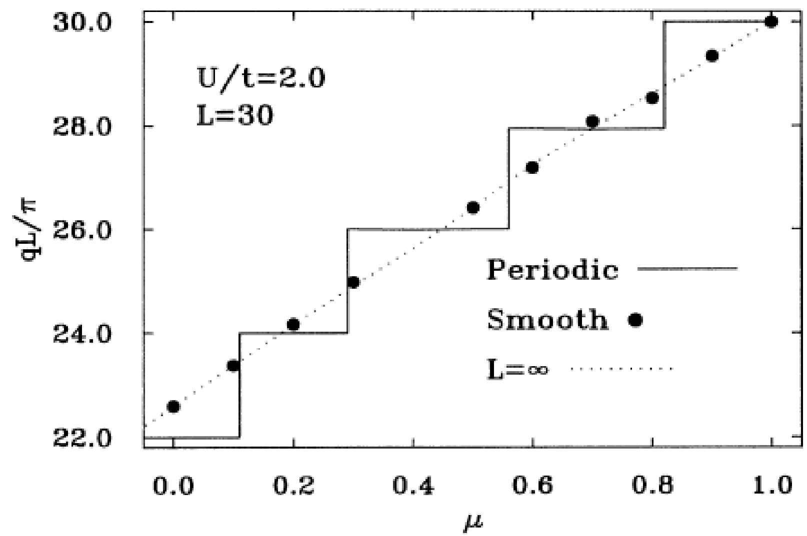


Smooth boundary conditions

1D S=1/2 Heisenberg



1D Hubbard

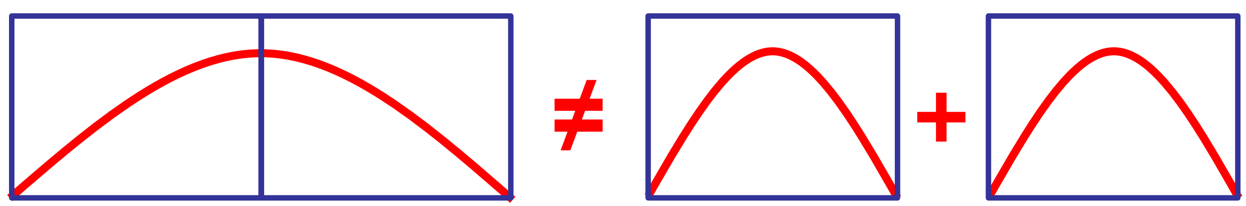


Succeed (to some extent)

to suppress boundary/finite-size effect

get smooth dependence on chemical potential

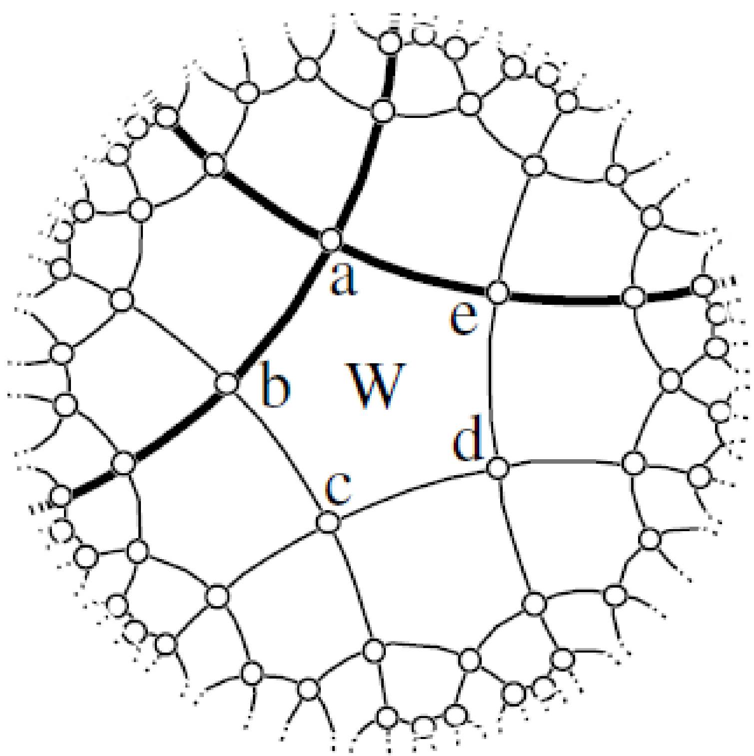
cf.) DMRG :
optimal mixture of
various boundary
condition using DM



◆ Statphys in curved space

classical model (e.g. Ising model $\mathcal{H} = J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$)

in **hyperbolic plane**

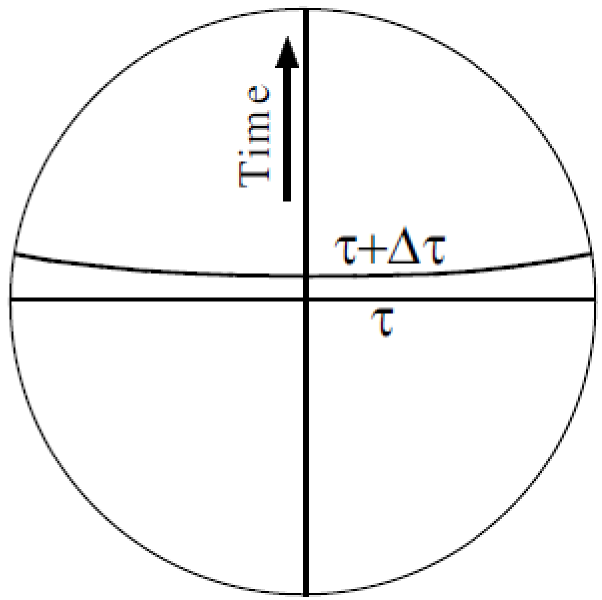


(5,4) tiling

◆ Hyperbolic deformation

Quantum mechanics
in 1+1 hyperbolic plane

Ueda et al. [PTP 124, 389 (2010)]



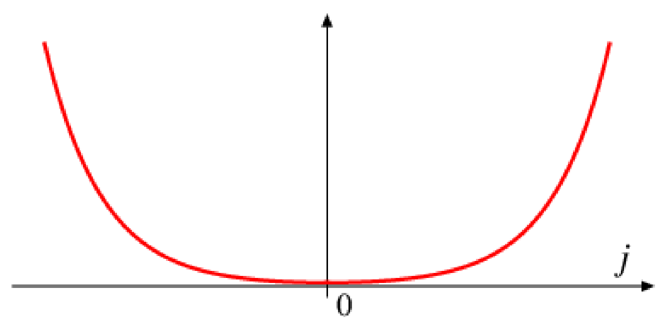
$$|\Psi(\tau + \Delta\tau)\rangle = \mathcal{U}(\Delta\tau)|\Psi(\tau)\rangle$$

$$\mathcal{U}(\Delta\tau) = \exp(-\Delta\tau\mathcal{H})$$

$$\mathcal{H} = \int \cosh(\nu x)\hat{h}(x)dx$$

Quantum model in 1D lattice

$$\mathcal{H}(\lambda) = \sum_{j=-L/2+1}^{L/2-1} \cosh(\lambda j)\, \hat{h}_{j,j+1}$$



Hyperbolic deformation

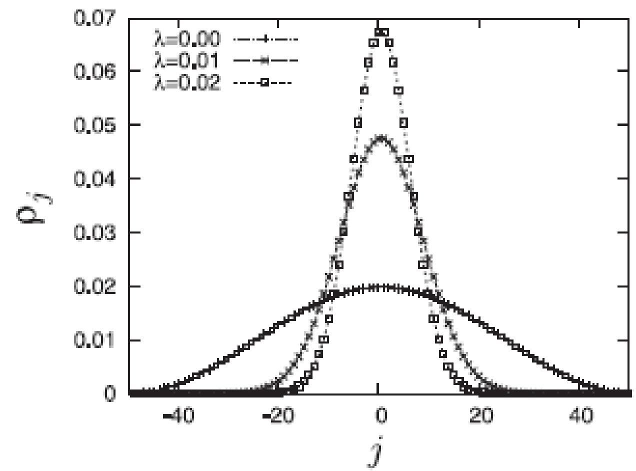
◆ Hyperbolic deformation

Hyperbolic deformation for 1D quantum spin systems

Ueda-Nishino [JPSJ 78, 014001 (2009)]
Ueda et al. [PTP 124, 389 (2010), JPSJ 80, 094001 (2011)]

Excited quasi-particle (in gapped system)
weakly confined
around the center of chain

→ avoid boundary effects

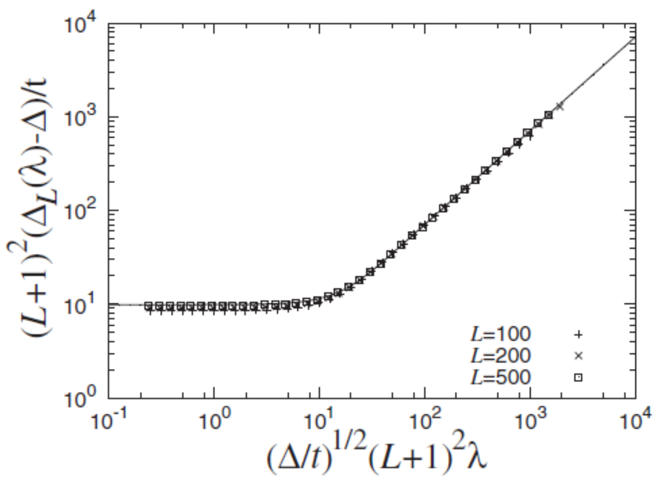


$$\mathcal{H}(L,\lambda) = \sum_{j=-L/2+1}^{L/2-1} \cosh(\lambda j)\, \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

(S=1)

Precise estimation
of excitation (Haldane) gap

$$\Delta = 0.410485$$



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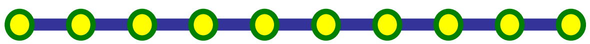
Smooth boundary condition in 1+1D system

1D quantum system → 1+1D classical system

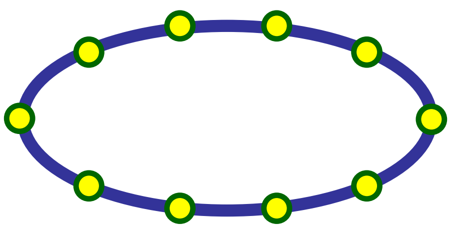
(periodic in imaginary-time direction)

open b.c. (uniform)

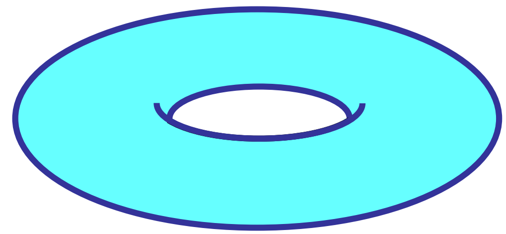


cylinder

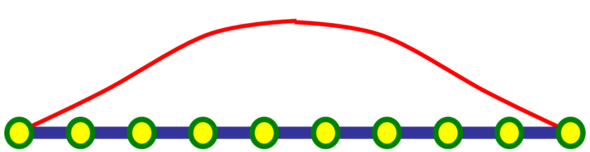
periodic b.c. (uniform)



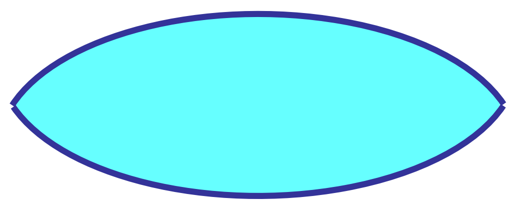
torus



smooth b.c.



closed surface



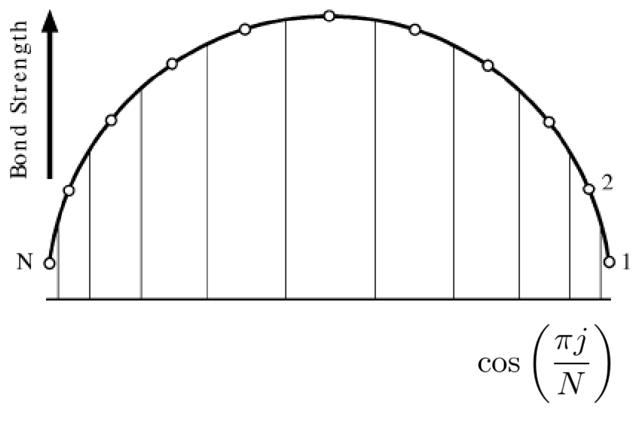
Spherical (sine) deformation

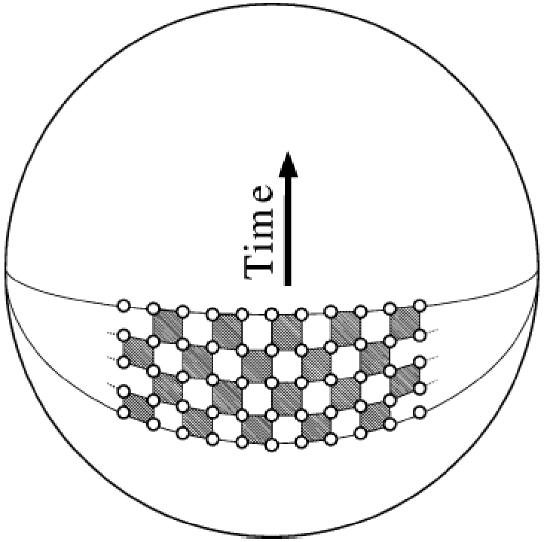
Spherical (sinusoidal) deformation

Gendiar-Krcmar-Nishino, PTP 122, 953 (2009)

$$\mathcal{H} = -t \sum_{j=1}^{N-1} \sin\left(\frac{\pi j}{N}\right) \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right)$$
$$-\mu \sum_{j=1}^{N} \sin\left(\frac{\pi(j-1/2)}{N}\right) c_j^\dagger c_j$$

Smooth turn-off
of energy scale of local Ham.
using sinusoidal function





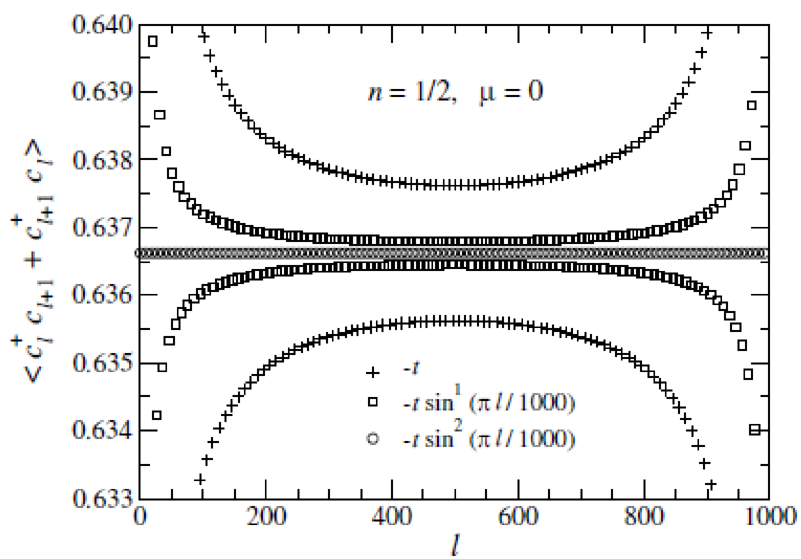
◆ Spherical (sine) deformation

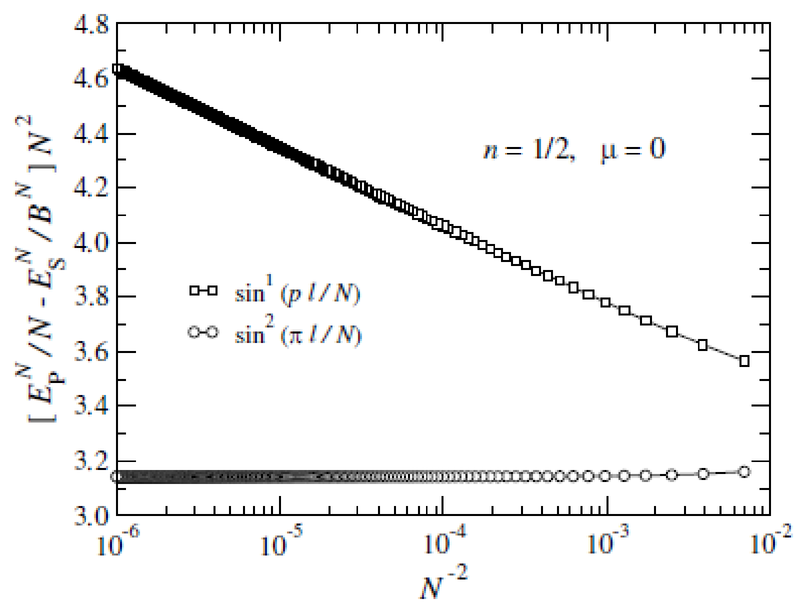
Sine-square deformation

Gendiar-Krcmar-Nishino, PTP 122, 953 (2009); 123, 393 (2010)

$$\mathcal{H} = -t \sum_{j=1}^{N-1} \sin^{2}\left(\frac{\pi j}{N}\right) \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right)$$

$$-\mu \sum_{j=1}^{N} \sin^{2}\left(\frac{\pi (j-1/2)}{N}\right) c_j^\dagger c_j$$





Complete suppression of boundary oscillation

Energy: Size scaling $\propto N^{-2}$

◆ Sine-Square Deformation

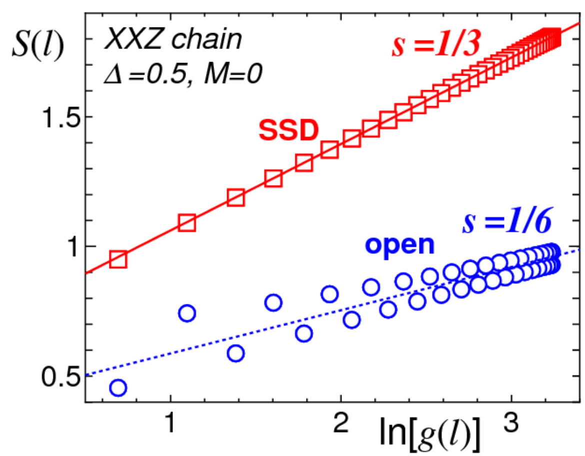
Gendiar et al.

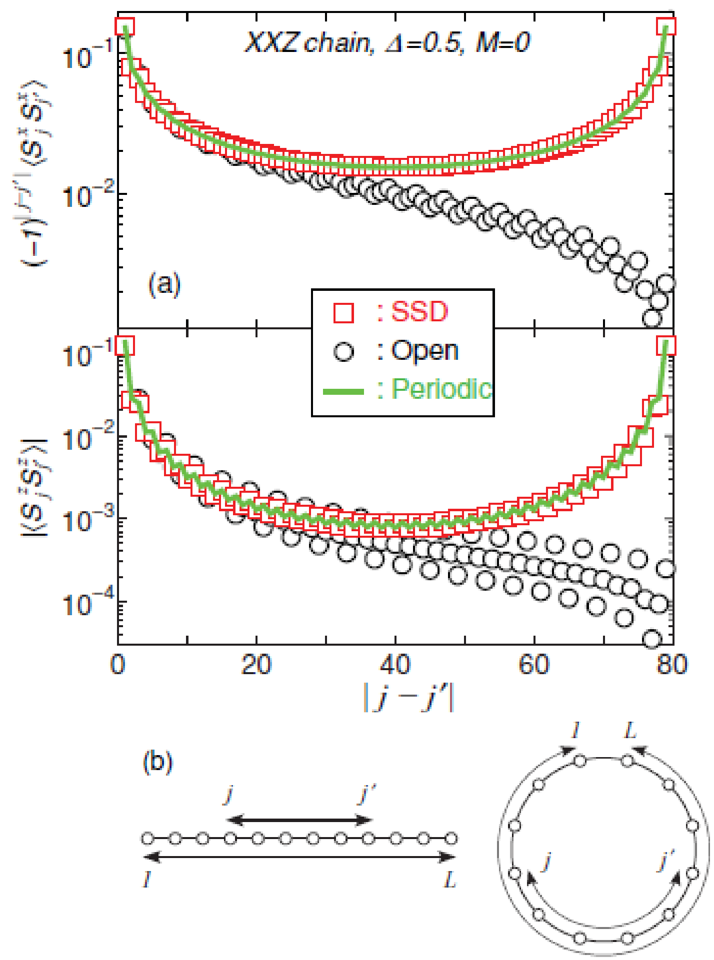
"In a certain sense, the system does not contain system boundary"

Sine-square deformation Hikihara-Nishino, PRB 83, 060414R (2011)

$$\mathcal{H} = \sum_{j=1}^{N-1} \sin^2\left(\frac{\pi j}{N}\right) \hat{h}_{j,j+1}$$

for various 1D spin system





Behaviors of correlation functions, Ent.Entropy, etc. completely coincide with those in periodic system

◆ Sine-Square Deformation

Sine-square deformation Hikihara-Nishino, PRB 83, 060414R (2011)

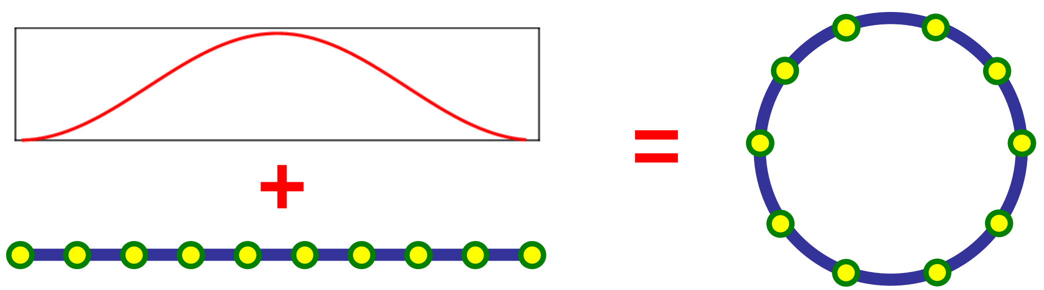
$$\mathcal{H} = \sum_{j=1}^{N-1} \sin^2\left(\frac{\pi j}{N}\right) \hat{h}_{j,j+1}$$

for various 1D spin system
in critical regime

Overlap between ground-state wave functions of
SSD and uniform-periodic systems

$$\langle v_{\mathrm{SSD}} | v_{\mathrm{PBC}} \rangle \simeq 1$$

the ground states of SSD and uniform periodic systems
are equivalent ***at level of wave function***

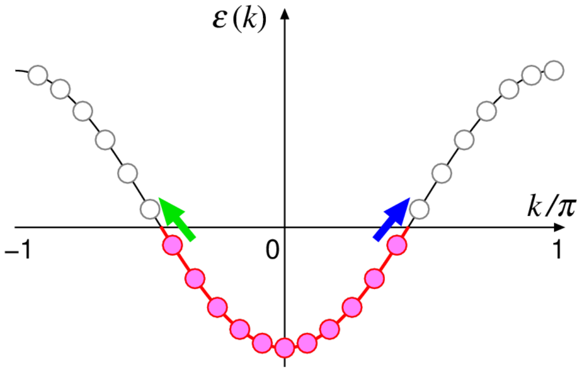


◆ Sine-Square Deformation

- numerical confirmation
  - free/interacting fermions [Gendiar et al. (2009, 2011)]
  - quantum spin systems at criticality [TH-Nishino (2011)]
  - Kondo-lattice model [Shibata-Hotta (2011)]

- rigorous proof
  - free/Dirac fermions, (anisotropic) XY spin chain, Gaussian model [Katsura (2011), Maruyama et al. (2011)]

- theory about mechanism of SSD
  - Fourier transform [Maruyama et al. (2011)]



- Field-theoretical description of SSD
  - CFT [Katsura (2012), Tada (2015), Ishibashi-Tada(2016) Okunishi(2016), Wen et al. (2016)]
  - SUSY QM [Okunishi-Katsura (2015)]

SSD Hamiltonian in k space

$$\mathcal{H} = -t\sum_j \sin^2\left(\frac{\pi j}{N}\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

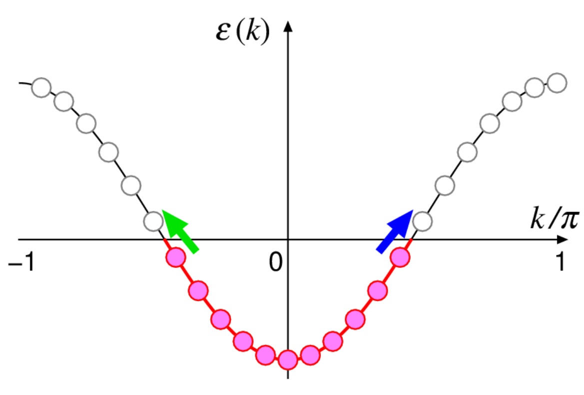
$$= -t\sum_j \left[1 - \frac{1}{2}\left(e^{i\frac{\pi j}{N}} + e^{i\frac{\pi j}{N}}\right)\right](c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

$$= \mathcal{H}_{\rm uni} + \frac{1}{2}\left[\mathcal{H}^{(+)}_{\rm chiral} + \mathcal{H}^{(-)}_{\rm chiral}\right]$$

Fourier transformation

$$\tilde{\mathcal{H}}_{\rm uni} = \sum_k \epsilon(k) c_k^\dagger c_k \qquad (\epsilon(k) = -2t\cos k)$$

$$\tilde{\mathcal{H}}^{(\pm)}_{\rm chiral} = e^{\mp i\delta/2}\sum_k \epsilon(k\mp\delta/2) c_k^\dagger c_{k\mp\delta}$$



hopping amp. is zero at Fermi points

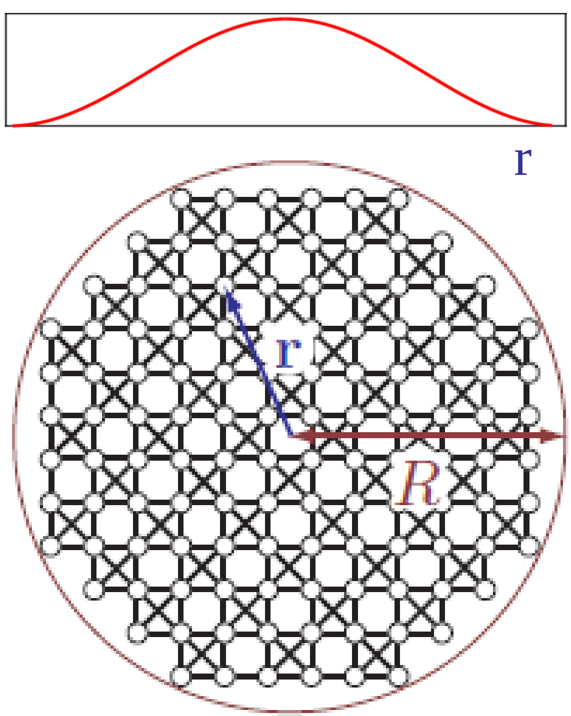
◆ Application of SSD

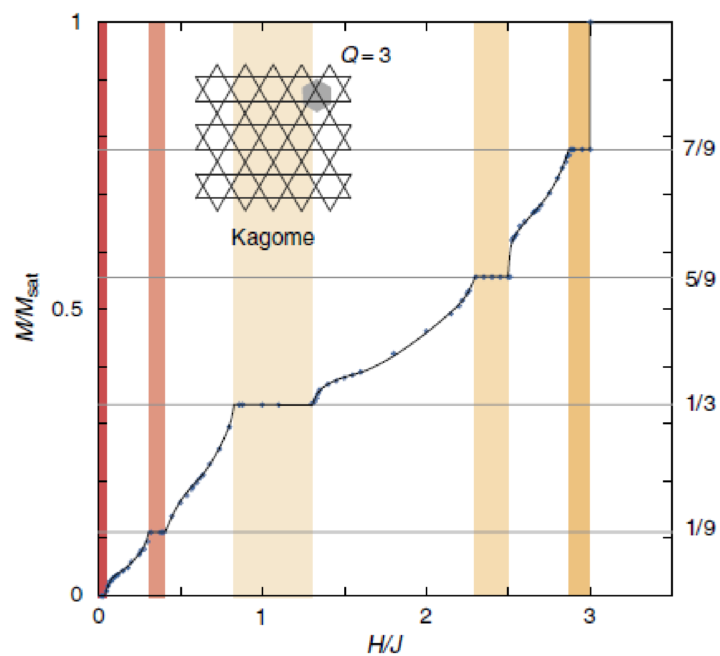
Grand-canonical SSD

[Hotta-Shibata PRB 86, 041108R (2012)
Hotta et al. PRB 87, 115129 (2013)]

Boundary regions of SSD system works as particle bath

Smoothing of magnetization curve
Suppression of boundary effects





[Nishimoto et al. Nat.Comm. (2013)]

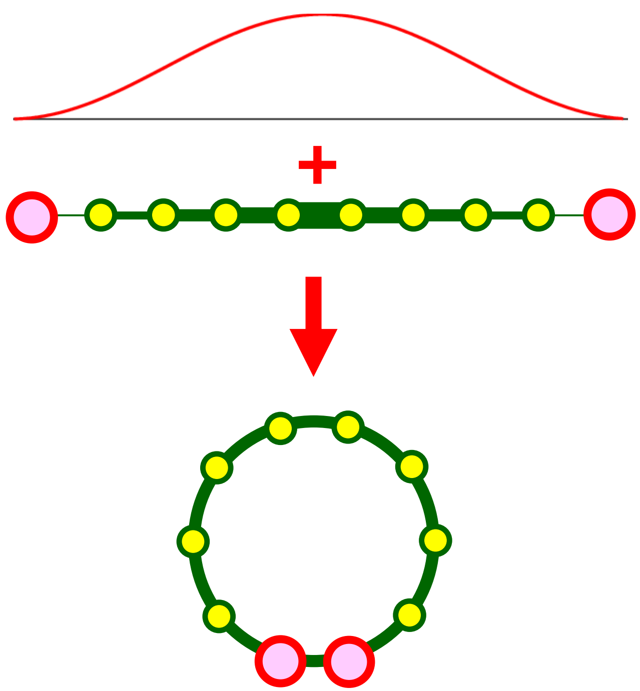
Applied to many frustrated magnets (Shibata group)

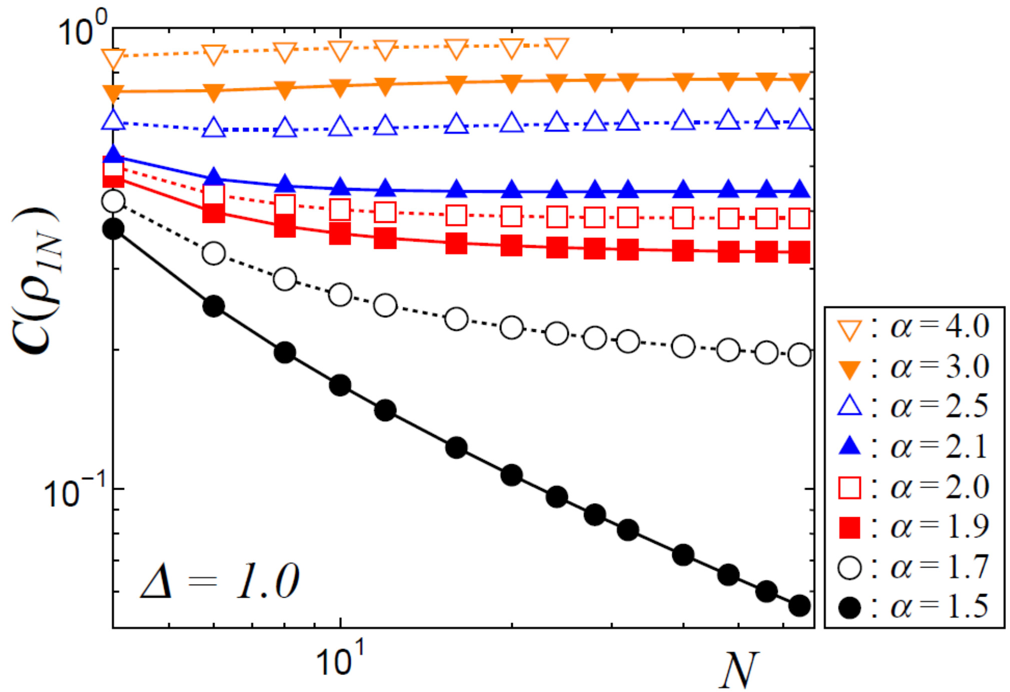
Application of SD-α

Long-Distance Entanglement in system with sin-α deformation (SD-α)

[Hikihara-Suzuki, PRA 87, 042337 (2013)]

$$\mathcal{H} = \sum_{j=1}^{N-1} \sin^{\alpha}\left(\frac{\pi j}{N}\right) \hat{h}_{j,j+1}$$

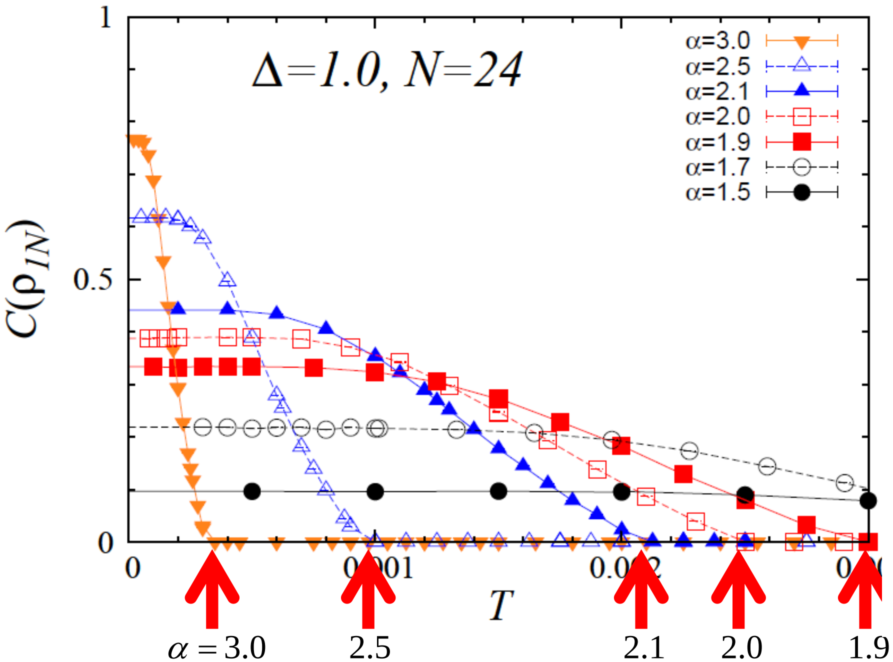


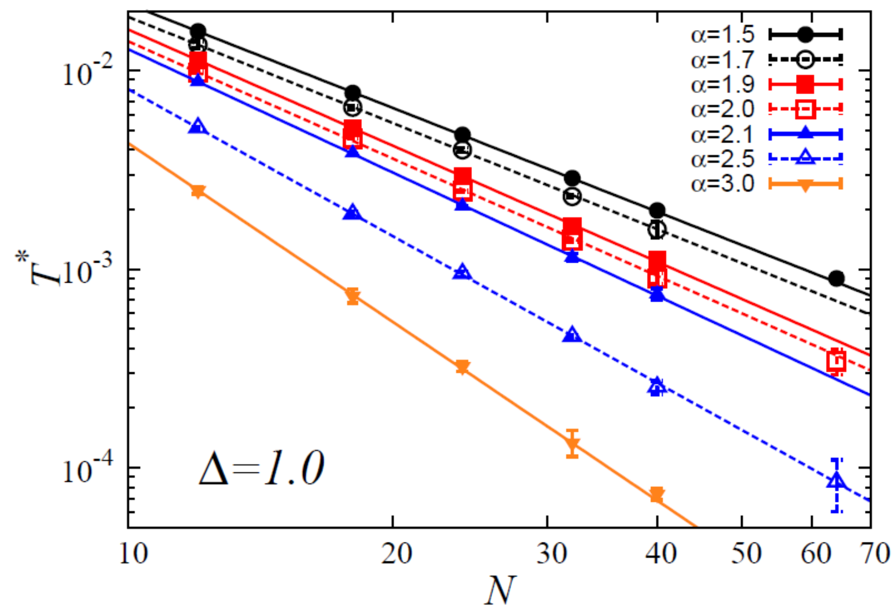


true LDE for $\alpha \geq 2$ in ground state

◆ SD-α at finite temperatures

critical temp. T^* at which LDE vanishes

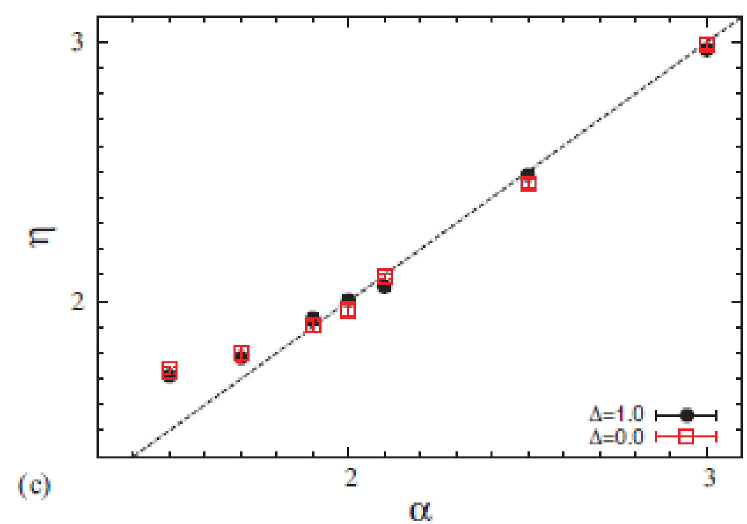




power-law decay $T^* \sim N^{-\eta}$

with $\eta = \alpha \quad (\alpha \geq 2)$

$\eta > \alpha \quad (\alpha < 2)$



cf: strength of edge bond

$$J_{\text{edge}} \sim \sin^{\alpha}\left(\frac{\pi}{N}\right) \sim N^{-\alpha}$$

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Summary of SSD

Smooth b.c.
tanh def.

physics in curved plane
hyperbolic def.

Spherical deformation
Sine-Square Deformation

Hamiltonian engineering
cold atom in optical lattice

Application
grand-canonical SSD

Field-theory
CFT

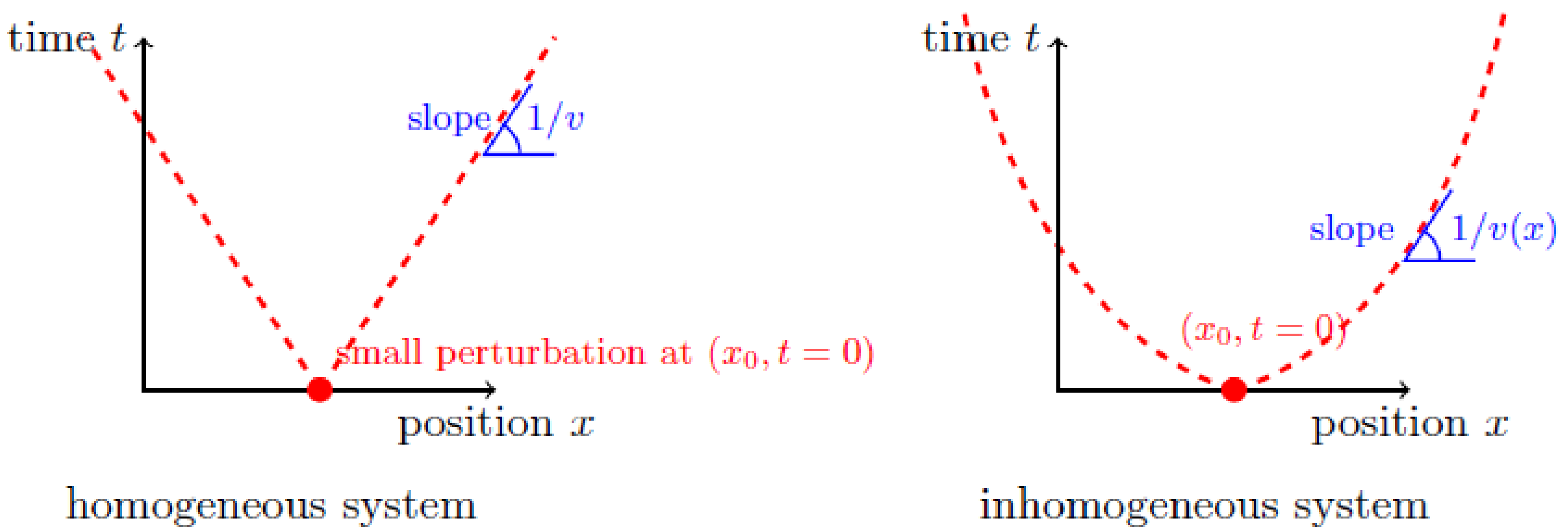
◆ Energy deformation in TLL

Tomonaga-Luttinger liquid [Dubail et al, arXiv1705.00679]

with uniform K and position-dependent velocity

$$\mathcal{H} = \sum_x f(x)\hat{h}_x \quad \longrightarrow \quad v(x) = vf(x)$$

$$ds^2 = dx^2 + [v(x)]^2 dt^2$$: curved light cone



Energy deformation in TLL

Tomonaga-Luttinger liquid with uniform K
and position-dependent velocity $v(x) = v f(x)$

$$\mathcal{H} = \sum_x f(x) \hat{h}_x$$

$$\tilde{\mathcal{H}} = \int dx \, v f(x) \left[\frac{1}{K} (\partial_x \theta)^2 + K (\partial_x \phi)^2 \right]$$

move to stretched coordinate: $\tilde{x} = \int_{x_0}^{x} \frac{du}{f(u)}$

$$\tilde{\mathcal{H}} = \int d\tilde{x} \, v \left[\frac{1}{K} (\partial_{\tilde{x}} \theta)^2 + K (\partial_{\tilde{x}} \phi)^2 \right]$$

"uniform" TLL