Sine-square deformations of 1D critical systems: exact results

桂 法称（東京大学・物理学専攻）

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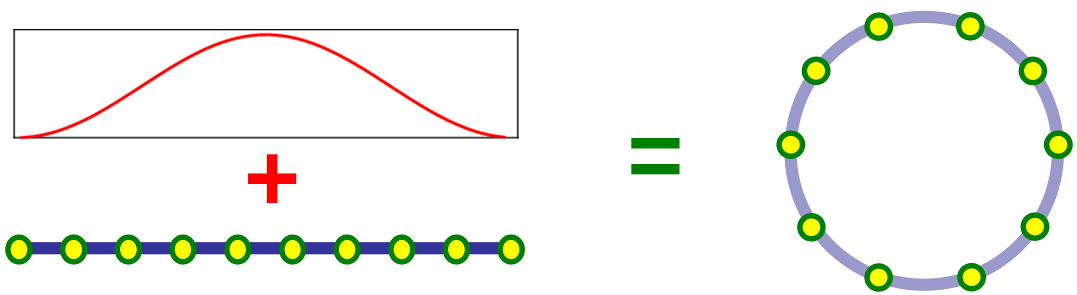

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Outline

1. Introduction

- What is SSD (sine-square deformation)?
- What is special about SSD?

2. Ground state of solvable models with SSD

- Definitions and properties
- Free fermion chain with SSD
- Other examples (spin chains, Dirac fermions, CFT, …)

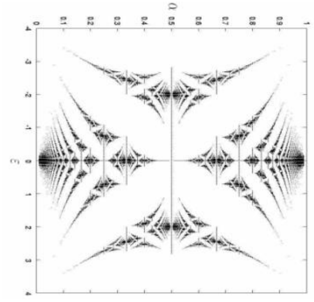
3. Excited states of solvable models with SSD

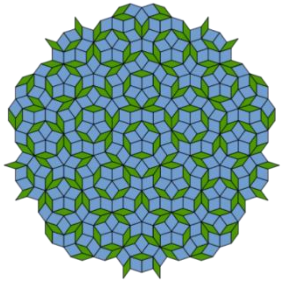
- What about excited states of SSD?
- Further steps towards exact solution

4. Summary

Many-body problem with inhomogeneities

- Disorder and inhomogeneity in (cond-mat) physics
  - Hofstadter butterfly, Wannier-Stark, …
  - Quasi-periodic systems
  - Impurity and boundary
    Anderson localization, Kondo problem, …





The presence of inhomogeneity and/or boundary
is usually an obstacle to solvability/integrability…

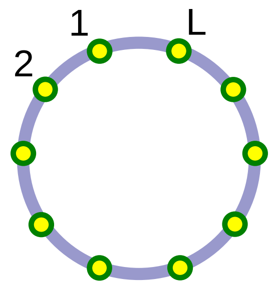
Main difficulty: Single-particle problem is already nontrivial.
What happens when the interaction is switched on?

- Today's talk
  - A new class of inhomogeneous ***but*** solvable models
    Accidentally discovered by numerics, but has a hidden CFT structure.
  - Abandon "from few to many" approach!
    Solve many-body problem ***without*** using single-particle solutions.

What is SSD (sine-square deformation)?

- Periodic chain

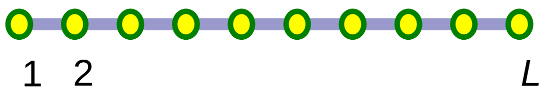
$$\mathcal{H}_0 = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



Any observable is translation invariant

- Open chain

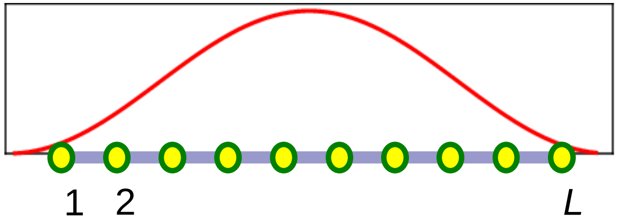
$$\mathcal{H}_{\rm open} = \sum_{j=1}^{L-1} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

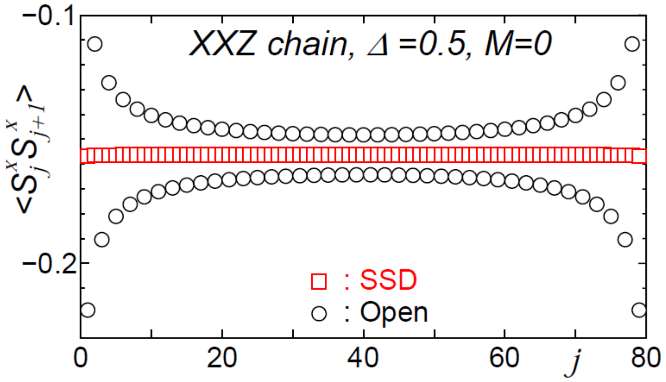


- Sine-square deformation

Gendiar *et al*., *PTP* (2009-2010), Hikihara & Nishino, *PRB* (2011).

$$\mathcal{H}_{\rm SSD} = \sum_{j=1}^{L-1} \sin^2(\pi j/L) \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$





No boundary effect in g.s.!

What is special about SSD?

- Suppression of boundary effects
  - Negligible Friedel oscillation, uniform g.s. correlations
  - Observed in ***1D critical systems***
    XXZ, Hubbard, Kondo-lattice (Shibata-Hotta, *PRB* (2011)), …

- Scaling of entanglement entropy

$$\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3}\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + s_1$$

$$\mathcal{S}^{\mathrm{OBC}}(\ell, L) = \frac{c}{6}\ln\left[\frac{2L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + \frac{s_1}{2} + \ln(g)$$

$$\mathcal{S}^{\mathrm{SSD}} \simeq \mathcal{S}^{\mathrm{PBC}}$$

- Wavefunction overlap

  Overlap between the g.s. of systems with PBC and SSD is almost 1.

$$\langle \Psi_{\mathrm{SSD}} | \Psi_{\mathrm{PBC}} \rangle \simeq 1$$

Conjecture.
G.S. of $\mathcal{H}_{\mathrm{SSD}}$ = G.S. of $\mathcal{H}_{\mathrm{PBC}}$

Main results:
☑ XY chain, Ising chain
☑ Massless Dirac, CFTs, …

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2. **Ground state of solvable models with SSD**

- Definitions and properties
- Free fermion chain with SSD
- Other examples (spin chains, Dirac fermions, CFT, …)

3. **Excited states of solvable models with SSD**

- What about excited states of SSD?
- Further steps towards exact solution

4. **Summary**

Definitions

- **Uniform and chiral Hamiltonians**

  Consider a lattice model on a chain of length *L*, or
  a continuous field theory on a ring of length ℓ. (PBC imposed)

	Lattice model	Field theory
Uniform	$\mathcal{H}_0 = \sum_{j=1}^{L} h_j + \sum_{j=1}^{L} h_{j,j+1}$	$\mathcal{H}_0 = \int_0^{\ell} h(x)\,dx$
Chiral	$\mathcal{H}_\pm = \sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} h_j + \sum_{j=1}^{L} e^{\pm i\delta j} h_{j,j+1}$, $\quad \delta = \frac{2\pi}{L}$	$\mathcal{H}_\pm = \int_0^{\ell} e^{\pm i\delta x} h(x)\,dx$, $\quad \delta = \frac{2\pi}{\ell}$

- **Sine-square deformed (SSD) Hamiltonian**

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

• Example: Heisenberg chain

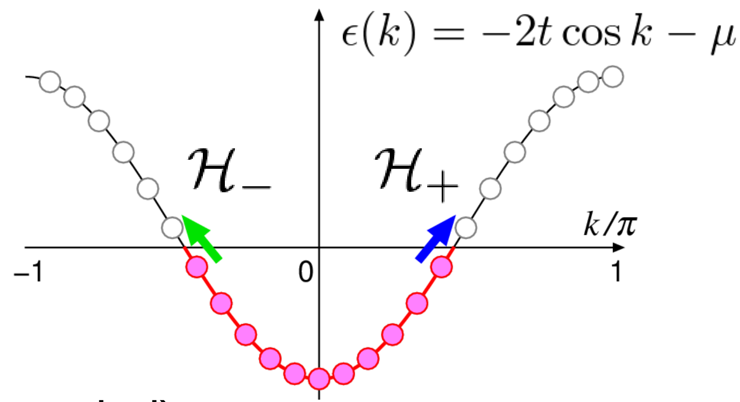
$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\sum_{j=1}^{L}\left(1 - \frac{1}{2}e^{i\delta j} - \frac{1}{2}e^{-i\delta j}\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1} = \sum_{j=1}^{L}\sin^2\left(\frac{\pi}{L}j\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$

Free fermion chain with SSD (1)

- Uniform and chiral Hamiltonians

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} c_j^\dagger c_j$$

$c_j / c_j^\dagger$: annihilation/creation of fermion at *j*.



Fourier.tr.

$$\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k$$

Ground state of $\mathcal{H}_0$: Fermi sea ($\epsilon(k) < 0$ occupied)

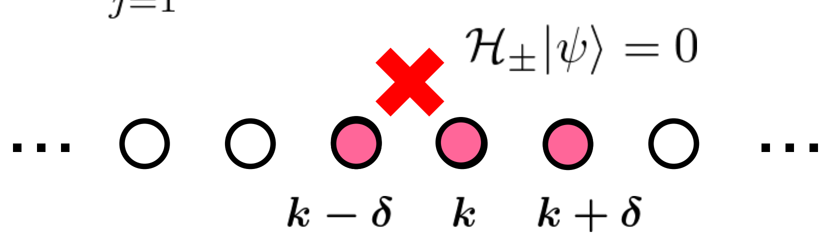
$$\mathcal{H}_0|\mathrm{FS}\rangle = E_g|\mathrm{FS}\rangle$$

Chiral Hamiltonian ($\delta = \frac{2\pi}{L}$)

$$\mathcal{H}_\pm = -t\sum_{j=1}^{L} e^{\pm i\delta j}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} c_j^\dagger c_j$$

Momentum rep.

$$\mathcal{H}_\pm = e^{\mp i\delta/2}\sum_k \epsilon(k \mp \delta/2) c_k^\dagger c_{k\mp\delta}$$



If $\epsilon(k_\mathrm{F} + \delta/2) = \epsilon(-k_\mathrm{F} - \delta/2) = 0$, then $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0.$ ($\because\ (c_k^\dagger)^2 = 0$)

Free fermion chain with SSD (2)

- SSD Hamiltonian

Real space

$$\mathcal{H}_{\mathrm{SSD}} = -t\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L-1}\sin^2\left[\frac{\pi}{L}\left(j-\frac{1}{2}\right)\right]c_j^\dagger c_j$$

In terms of $\mathcal{H}_0$ & $\mathcal{H}_\pm$, $\quad \mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$

Fermi sea is annihilated by chiral Hamiltonians! $\boxed{\mathcal{H}_\pm|\mathrm{FS}\rangle = 0}$

$$\mathcal{H}_{\mathrm{SSD}}|\mathrm{FS}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \cancel{\frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)}\right]|\mathrm{FS}\rangle = \frac{E_g}{2}|\mathrm{FS}\rangle$$

Fermi sea is an exact eigenstate of $\mathcal{H}_{\mathrm{SSD}}$!

- Uniqueness of the ground state

Fermi sea is ***the unique*** g.s. of $\mathcal{H}_{\mathrm{SSD}}$. $\mathcal{H}_0$ & $\mathcal{H}_{\mathrm{SSD}}$ share the same g.s.

Proof. Free-fermion chain → XY spin chain (via Jordan-Wigner tr.)

Perron-Frobenius theorem tells: (i) the ground state of $\mathcal{H}_{\mathrm{SSD}}$ is unique.
(ii) it has nonvanishing overlap with $|\mathrm{FS}\rangle$, the ground state of $\mathcal{H}_0$.

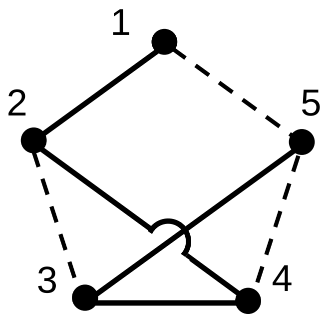
Application of Perron-Frobenius theorem

Theorem (Perron-Frobenius).
Let A be an $N \times N$ real symmetric matrix with the properties
(i) $a_{i,j} \leq 0$ for any $i \neq j$,
(ii) all $i \neq j$ are connected via nonzero matrix elements of A.
Then the lowest eigenvalue of A is nondegenerate and
the corresponding eigenvector $\boldsymbol{v} = (v_1, \ldots, v_N)$
can be taken to satisfy $v_i > 0$ for all i.

Connectivity condition

$\mathcal{H}_{\mathrm{SSD}}$ satisfy both
conditions (i) and (ii).
→ unique g.s.

$$\begin{pmatrix} & * & 0 & 0 & 0 \\ * & & 0 & * & 0 \\ 0 & 0 & & * & * \\ 0 & * & * & & 0 \\ 0 & 0 & * & 0 & \end{pmatrix}$$



$|\mathrm{FS}\rangle$ (in spin reps.) which is an eignstate of $\mathcal{H}_{\mathrm{SSD}}$
can also be taken to satisfy $v_i > 0$ for all i.
This state cannot be orthogonal to the SSD g.s.
→ $|\mathrm{FS}\rangle$ is the unique ground state of $\mathcal{H}_{\mathrm{SSD}}$.

Real-space picture

- **Determinant identity**

1-particle eigenstates of $\mathcal{H}_0$: $\phi_k(j) = e^{\mathbf{i}kj}$ (plane waves)
1-particle eigenstates of $\mathcal{H}_{\rm SSD}$: $\psi_k(j) = ?$

Their Slater determinants are identical when
the states are occupied up to the Fermi level E_F.

$$\det[\psi_k(j)]_{k,j=1,\ldots,N} = \det[\phi_k(j)]_{k,j=1,\ldots,N}$$

Solved many-body problem ***without*** using 1-particle solutions!

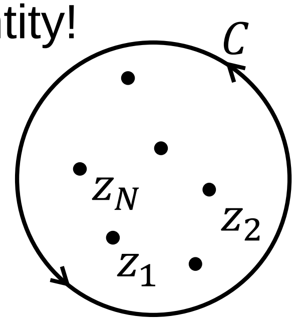
- **Curious identity** (H.K., *JPA* **44**, 252001 (2011))

First quantization picture: Fermi sea = Vandermonde det.

$$\Delta(z_1,\ldots,z_N) = \prod_{1\le i<j\le N}(z_i - z_j) \qquad z_j = \exp\left(i\frac{2\pi}{L}x_j\right)$$

$\mathcal{H}_\pm|\mathrm{FS}\rangle = 0$ translates into a remarkable identity!
For any set of $\{z_1, \ldots, z_N\}$ and t, we have

$$\sum_{j=1}^{N} z_j \prod_{k(\neq j)} \frac{z_j - t z_k}{z_j - z_k} = \sum_{j=1}^{N} z_j$$



$$f(z) = \prod_{1\le j\le N} \frac{z - t z_j}{z - z_j}$$

Anisotropic XY chain

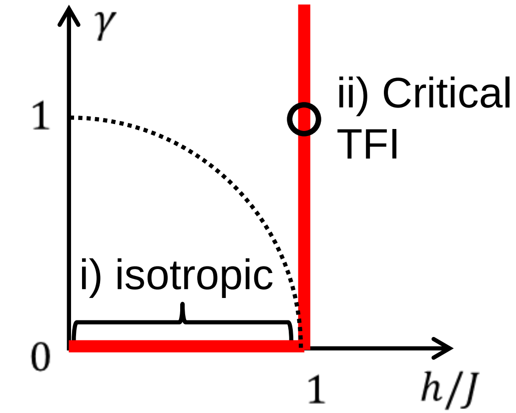
■ Uniform Hamiltonian

$$\mathcal{H}_0 = -J\sum_{j=1}^{L}[(1+\gamma)S_j^x S_{j+1}^x + (1-\gamma)S_j^y S_{j+1}^y] - h\sum_{j=1}^{L} S_j^z,$$

$$\Longrightarrow \quad \mathcal{H}_0 = \sum_{k\in\mathcal{K}} \epsilon_0(k)\left(d_k^\dagger d_k - \frac{1}{2}\right)$$

Jordan-Wigner,
Fourier, Bogoliubov. tr.

Phase diagram



Ground state of $\mathcal{H}_0$: $d_k|0\rangle = 0$ for all *k*.

■ Chiral and SSD Hamiltonians

Chiral Hamiltonian ($\delta = \frac{2\pi}{L}$) in momentum space

$$\mathcal{H}_\pm = \frac{1}{2}e^{\mp i\delta/2}\sum_{k\in\mathcal{K}}\left[\epsilon_\pm(k)d_k^\dagger d_{k\mp\delta} - \cancel{i\eta_\pm(k)d_k^\dagger d_{-k\pm\delta}^\dagger} + i\eta_\pm(k)d_{-k}d_{k\mp\delta} - \epsilon_\pm(k)d_{-k}d_{k\mp\delta}^\dagger,\right]$$

$\eta_\pm(k) = 0$ for all *k* when i) $\gamma = 0$, ii) $\gamma = 1,\ h/J = 1$, in which case $\boxed{\mathcal{H}_\pm|0\rangle = 0}$

$$\Longrightarrow \quad \mathcal{H}_{\rm SSD}|0\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|0\rangle = \frac{E_g}{2}|0\rangle$$

$|0\rangle$ is ***the unique*** ground state of $\mathcal{H}_{\rm SSD}$ (Perron-Frobenius theorem).

Critical Potts chain

- **Uniform Hamiltonian**

  Z_3 Pauli operators
  $$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

  $$H_0 = -f \sum_{j=1}^{L} (\tau_j + \tau_j^\dagger) - J \sum_{j=1}^{L} (\sigma_j^\dagger \sigma_{j+1} + \sigma_j \sigma_{j+1}^\dagger)$$

  PBC: $\sigma_{L+1} = \sigma_1$

  The model is critical (c=4/5), self-dual, and integrable when f=J. However, it is not reducible to free fermions. (parafermions?)

- **SSD Hamiltonian**

  $$H_{\mathrm{SSD}} = -f \sum_{j=1}^{L} \sin^2\left(\frac{\pi(j-1/2)}{L}\right) (\tau_j + \tau_j^\dagger) - J \sum_{j=1}^{L-1} \sin^2\left(\frac{\pi j}{L}\right) (\sigma_j^\dagger \sigma_{j+1} + \sigma_j \sigma_{j+1}^\dagger)$$

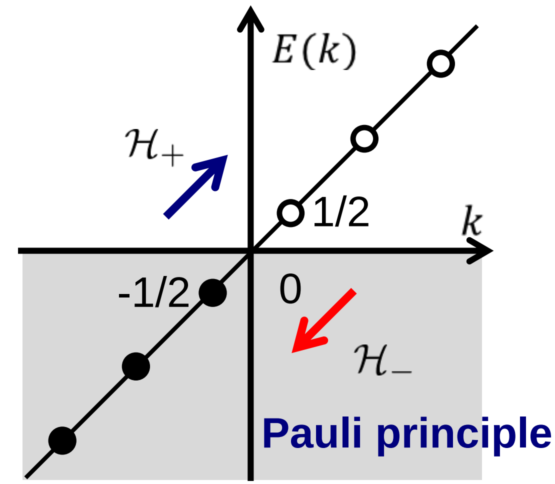
- **Numerical result**

  At the critical point (f=J), the overlap between the g.s. of uniform and SSD Hamiltonians is remarkably close to 1!

  $$1 - \langle \psi_{\mathrm{SSD}} | \psi_{\mathrm{PBC}} \rangle \sim 10^{-5}$$

  Numerical diagonalization by *Mathematica* up to 16 sites (3^{16}= 43,046,721).

Massless Dirac fermions



- Uniform Hamiltonian

Ring of length ℓ. APBC: $\psi_R(x+\ell) = -\psi_R(x)$

$$\mathcal{H}_0 = -i\frac{v_{\mathrm{F}}}{2\pi}\int_0^\ell dx\ :\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x):$$

Fourier.tr. $\Rightarrow$ $$\mathcal{H}_0 = \frac{2\pi}{\ell}v_{\mathrm{F}}\sum_{n\in\mathbb{Z}+\frac{1}{2}} n:\psi_{R,n}^\dagger\psi_{R,n}:$$

Ground state of $\mathcal{H}_0$: Dira sea (*E*<0 occupied) $\mathcal{H}_0|\mathrm{DS}\rangle = 0$

- Chiral and SSD Hamiltonians

$(\mathcal{H}_\pm)^\dagger = \mathcal{H}_\mp$

$$\mathcal{H}_\pm = \ -i\frac{v_{\mathrm{F}}}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x}:\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x):\pm\frac{\pi v_{\mathrm{F}}}{2\ell}\frac{1}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x}:\psi_R^\dagger(x)\psi_R(x):$$

$\Rightarrow$ $$\mathcal{H}_\pm = \frac{2\pi}{\ell}v_{\mathrm{F}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(n\pm\frac{1}{2}\right)\psi_{R,n\pm1}^\dagger\psi_{R,n}$$ $$\boxed{\mathcal{H}_\pm|0\rangle = 0}$$

Dirac sea $|\mathrm{DS}\rangle$ is ***a*** ground state of $\mathcal{H}_{\mathrm{SSD}}$.

Proof. $\mathcal{H}_{\mathrm{SSD}}$ is positive semi-definite ($\langle\Psi|\mathcal{H}_{\mathrm{SSD}}|\Psi\rangle \geq 0$ for any state),

which follows from $$\mathcal{H}_{\mathrm{SSD}} = \sum_{n\in\mathbb{Z}+\frac{1}{2},n>0}(\alpha_{R,n}^\dagger\alpha_{R,n}+\beta_{R,n}\beta_{R,n}^\dagger)$$

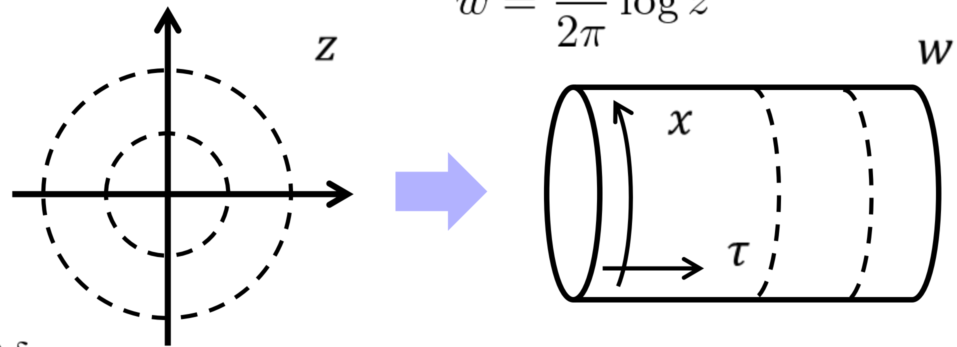
$$\alpha_{R,n} = \psi_{R,n}-\psi_{R,n+1}$$
$$\beta_{R,n} = \psi_{R,-n}-\psi_{R,-n-1}$$

(1+1) d Conformal field theories

- **Uniform Hamiltonian**

$$\mathcal{H}_0 = \int_0^{\ell} \frac{dx}{2\pi}\left(T_{\rm cyl}(w) + \overline{T}_{\rm cyl}(\overline{w})\right)$$

$$w = \frac{\ell}{2\pi}\log z$$



Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Ground state: vacuum state, $|{\rm vac}\rangle = |0\rangle \otimes |\bar{0}\rangle \quad (L_n|0\rangle = 0,\ n \geq -1)$

- **Chiral and SSD Hamiltonians** ($\delta = \frac{2\pi}{L}$)

$$\mathcal{H}_\pm = \int_0^{\ell} \frac{dx}{2\pi}\left(e^{\pm\delta w} T_{\rm cyl}(w) + e^{\mp\delta\overline{w}}\,\overline{T}_{\rm cyl}(\overline{w})\right)$$

$$\mathcal{H}_{\rm SSD} = \mathcal{H}_L + \mathcal{H}_R - \frac{\pi c}{12\ell} \qquad \mathcal{H}_L = \frac{\pi}{\ell}\left(L_0 - \frac{L_{+1} + L_{-1}}{2}\right)$$

Finite-size scaling of the g.s. energy

- **Vacuum state**

$|0\rangle$ is an E=0 eigenstate of $\mathcal{H}_{\rm SSD}$. (SL(2,**C**) invariance: $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$)

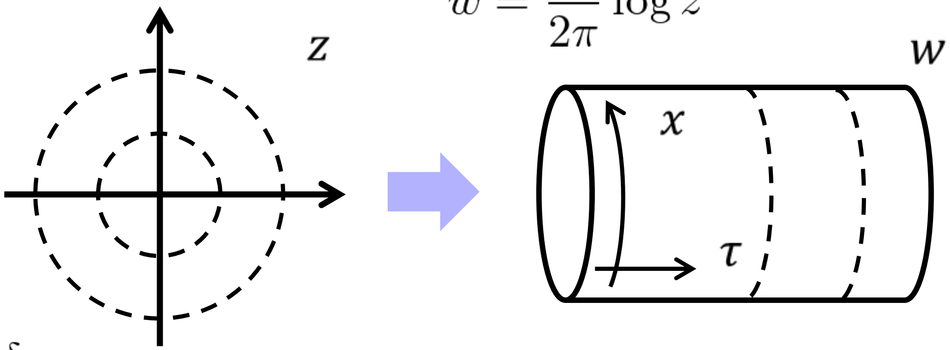
Moreover, $|0\rangle$ is the unique ground state of $\mathcal{H}_{\rm SSD}$ for unitary CFTs.

(1+1) d Conformal field theories

■ Uniform Hamiltonian

$$\mathcal{H}_0 = \frac{2\pi}{\ell}(L_0 + \overline{L}_0) - \frac{\pi c}{6\ell}$$

$$w = \frac{\ell}{2\pi}\log z$$



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■ Chiral and SSD Hamiltonians ($\delta = \frac{2\pi}{L}$)

$$\mathcal{H}_\pm = \frac{2\pi}{\ell}(L_{\pm 1} + \overline{L}_{\mp 1})$$

$$\mathcal{H}_{\mathrm{SSD}} = \mathcal{H}_L + \mathcal{H}_R - \frac{\pi c}{12\ell} \qquad \mathcal{H}_L = \frac{\pi}{\ell}\left(L_0 - \frac{L_{+1} + L_{-1}}{2}\right)$$

Finite-size scaling of the g.s. energy

■ Vacuum state

$|0\rangle$ is an E=0 eigenstate of $\mathcal{H}_{\mathrm{SSD}}$. (*SL*(2,**C**) invariance: $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$)

Moreover, $|0\rangle$ is the unique ground state of $\mathcal{H}_{\mathrm{SSD}}$ for unitary CFTs.

Positivity of H_{SSD}

- c=1 CFT (Free-boson theory)
  - Bosonization of Virasoro generators

$$L_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} : a_m a_{n-m} :$$

Heisenberg algebra

$$[a_n, a_m] = n\delta_{n+m,0}$$

  - H_{SSD} in terms of a's (charged bosonic Fock space) $a_n^\dagger = a_{-n}$

$$\mathcal{H}_L = \frac{\pi}{2\ell}\sum_{n\geq 0}(a_{-n} - a_{-n-1})(a_n - a_{n+1})$$

Positive semi-definite

  - H_{SSD} in real space

$$\mathcal{H}_L + \mathcal{H}_R = \int_0^\ell dx\, f(x) : \left(\frac{d\phi_L}{dx}\right)^2 + \left(\frac{d\phi_R}{dx}\right)^2 :, \qquad f(x) = \sin^2\left(\frac{\pi x}{\ell}\right)$$

- CFT associated with Affine-Lie algebra $c = \dfrac{k\,\dim\mathfrak{g}}{k+h^\vee}$
  - Sugawara construction

$$L_n = \frac{1}{2(k+h^\vee)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m\in\mathbb{Z}} : J_m^a J_{n-m}^a : \qquad [J_n^a, J_m^b] = i f_c^{ab} J_{n+m}^c + kn\delta^{ab}\delta_{n+m,0}$$

$$\mathcal{H}_{\mathrm{L}} = \frac{\pi}{2\ell(k+h^\vee)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{n\geq 0}(J_{-n}^a - J_{-n-1}^a)(J_n^a - J_{n+1}^a)$$

Positive semi-definite

Vacuum $|0\rangle$ is a ground state of $\mathcal{H}_{\mathrm{SSD}}$. ***Is it unique?***

$\mathcal{N}$=2 SCFT

$J(z)$: Δ=1 U(1) current, $G^{\pm}(z)$: Δ=3/2 fermionic fields

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$
$$[L_m, G^{\pm}_r] = \left(\frac{m}{2} - r\right) G^{\pm}_{m+r},$$
$$[L_m, J_n] = -nJ_{m+n},$$
$$[J_m, G^{\pm}_r] = \pm G^{\pm}_{m+r},$$
$$\{G^{\pm}_r, G^{\mp}_s\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},$$
$$\{G^{+}_r, G^{+}_s\} = \{G^{-}_r, G^{-}_s\} = 0,$$
$$[J_m, J_n] = \frac{c}{3} m \delta_{m+n,0},$$

$$m, n \in \mathbb{Z},$$
$$r, s \in \mathbb{Z} + \alpha$$
$$\alpha = 0 \quad (\mathrm{Ramond})$$
$$\alpha = \frac{1}{2} \quad (\mathrm{Neveu-Schwarz})$$

- H_0 in Ramond sector

$$\frac{2\pi}{\ell}\left(L_0 - \frac{c}{24}\right) = \frac{\pi}{\ell}\{G^{+}_0, G^{-}_0\}$$

- H_{SSD} in Neveu-Schwarz sector

$$\mathcal{H}_{\mathrm{L}} = \frac{\pi}{\ell}\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) = \frac{\pi}{2\ell}\{Q, Q^{\dagger}\}$$

Supersymmetry

$$Q^{\dagger} = \frac{G^{+}_{\frac{1}{2}} - G^{+}_{-\frac{1}{2}}}{\sqrt{2}}$$
$$Q^2 = (Q^{\dagger})^2 = 0$$

- 1-parameter family connecting R and NS (spectral flow)?

Lattice models ⇔ Field theories

- Koo-Saleur approach

W.M. Koo and H. Saleur, *NPB* **426**, 459 (1994).

$e_j \quad (j = 1, 2, ..., 2L)$: Generators of Temperley-Lieb algebra

$$\mathcal{H}_0 = -\frac{\gamma}{\pi \sin\gamma} \sum_{j=1}^{2L} (e_j - \text{const.}) \qquad \mathcal{P}_0 = \frac{\gamma}{i\pi \sin\gamma} \sum_{j=1}^{2L} [e_j, e_{j+1}]$$

Conjecture:

$$L_n^{(\text{lattice})} := \frac{L}{2\pi} \left\{ -\frac{\gamma}{\pi \sin\gamma} \sum_{j=1}^{2L} e^{inj\pi/L} \left(e_j + \frac{i\gamma}{\pi \sin\gamma} [e_j, e_{j+1}] - \text{const.} \right) \right\} + \frac{c}{24} \delta_{n,0}$$

gives the Virasoro generators in the scaling limit.

- Revival

- A. Milsted and G. Vidal, arXiv:1706.01436
  Applying the formula to nonintegrable models.

$$c \simeq 2\langle I | H_2^\dagger H_2 | I \rangle$$

$$H_n = \frac{N}{2\pi} \sum_{j=1}^{N} e^{inj\,2\pi N}\, h_j$$

- M.S. Zin and Z.Wang, arXiv:1706.08497
  Towards a rigorous proof of Koo-Saleur formula.

Towards a rigorous proof of SSD Hamiltonian in CFT from lattice?

Outline

1. Introduction

- What is SSD (sine-square deformation)?
- What is special about SSD?

2. Ground state of solvable models with SSD

- Definitions and properties
- Free fermion chain with SSD
- Other examples (spin chains, Dirac fermions, CFT, …)

3. Excited states of solvable models with SSD

- What about excited states of SSD?
- Further steps towards exact solution

4. Summary

What about excited states of SSD?

- Gapped or gapless?
  - Lieb-Schultz-Mattis argument (LSM, *Ann. Phys*. **16** (1961)

G.S. of H_{SSD} (XY-chain) Trial state

$$|\Psi_0\rangle \quad \Rightarrow \quad |\Psi_1\rangle := U|\Psi_0\rangle, \quad U = \exp\left(i\sum_{j=1}^{L}\frac{2\pi}{L}jS_j^z\right)$$

Orthogonality

$|\Psi_0\rangle$ is translation invariant.

$$\begin{aligned}\langle\Psi_0|\Psi_1\rangle &= \langle\Psi_0|U|\Psi_0\rangle \\ &= \langle\Psi_0|TUT^{-1}|\Psi_0\rangle = -e^{-2\pi iM/L}\langle\Psi_0|\Psi_1\rangle\end{aligned}$$

$$\langle\Psi_0|\Psi_1\rangle = 0$$

if M (magnetization) is not $\pm L/2$.

Upper bound of gap

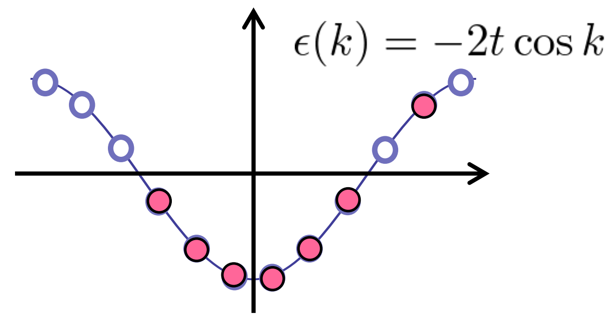
$$\begin{aligned}\Delta E &= \langle\Psi_1|H_{\mathrm{SSD}}|\Psi_1\rangle - \langle\Psi_0|H_{\mathrm{SSD}}|\psi_0\rangle \\ &= \langle\Psi_0|U^\dagger H_{\mathrm{SSD}}U - H_{\mathrm{SSD}}|\Psi_0\rangle \\ &\leq \frac{\pi^2 J}{L} + O(1/L^2)\end{aligned}$$

because $\sin^2(\pi j/L)$ is $O(1)$ for all j.

This suggests that 1d critical system with SSD is still critical.
But is the upper bound really optimal? → **NO!**

Free-fermion chain with SSD

- Hamiltonian (reminder)



$$\mathcal{H}_0 = -t \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)\,.$$

$$\mathcal{H}_{\mathrm{SSD}} = -t \sum_{j=1}^{L-1} \sin^2\left(\frac{\pi}{L} j\right) (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)\,.$$

Ansatz state: $|\Psi\rangle = \left(\sum_{k\in\circ} \psi_k c_k^\dagger\right) |\mathrm{FS}\rangle$ Fermi sea + one extra fermion

Using $(\mathcal{H}_{\mathrm{SSD}} - E_g/2)|\mathrm{FS}\rangle = 0$,
we get Harper-like eq. in *k*-space (*m*=0,1,…,*L*/2-1):

$$-\sin\left(\frac{2\pi}{L}m\right)\psi_{m-1} + 2\sin\left[\frac{2\pi}{L}\left(m+\frac{1}{2}\right)\right]\psi_m - \sin\left[\frac{2\pi}{L}(m+1)\right]\psi_{m+1} = \varepsilon\psi_m$$

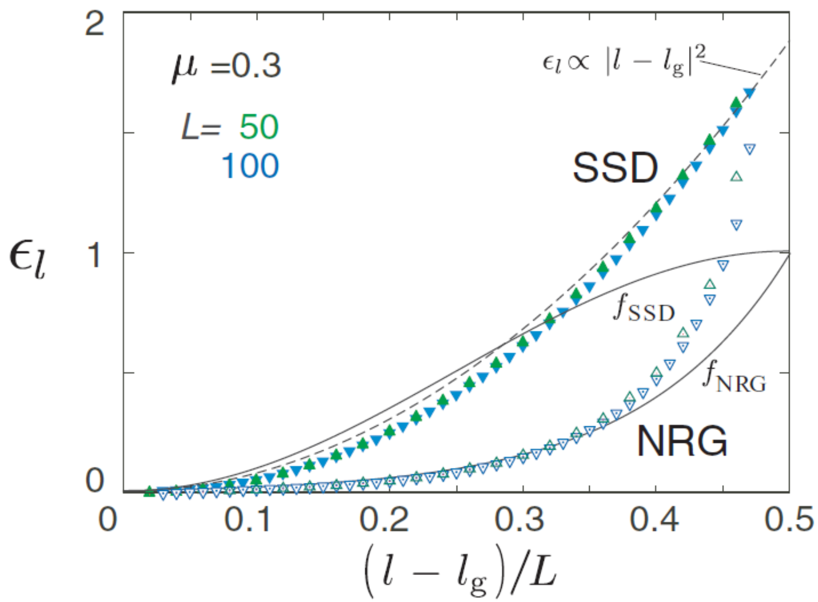
$$\forall \psi_m = \sqrt{\frac{2}{L}}$$

- Scaling of excitation energy

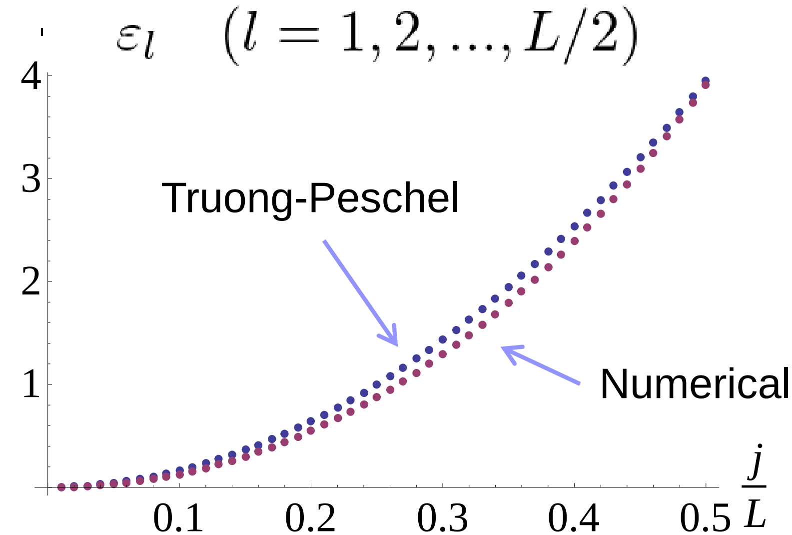
A simple variational argument shows that $\varepsilon \leq \frac{2\pi}{L^2} + O\left(\frac{1}{L^3}\right)$.
Very low-energy states!
Usual CFT scaling (1/*L*) breaks down.

Scaling property of excitation energy

Hotta *et al.*, *PRB* (2013)
μ=0.3, L=50 & 100.



Numerical v.s. analytical
μ=0, L=100.

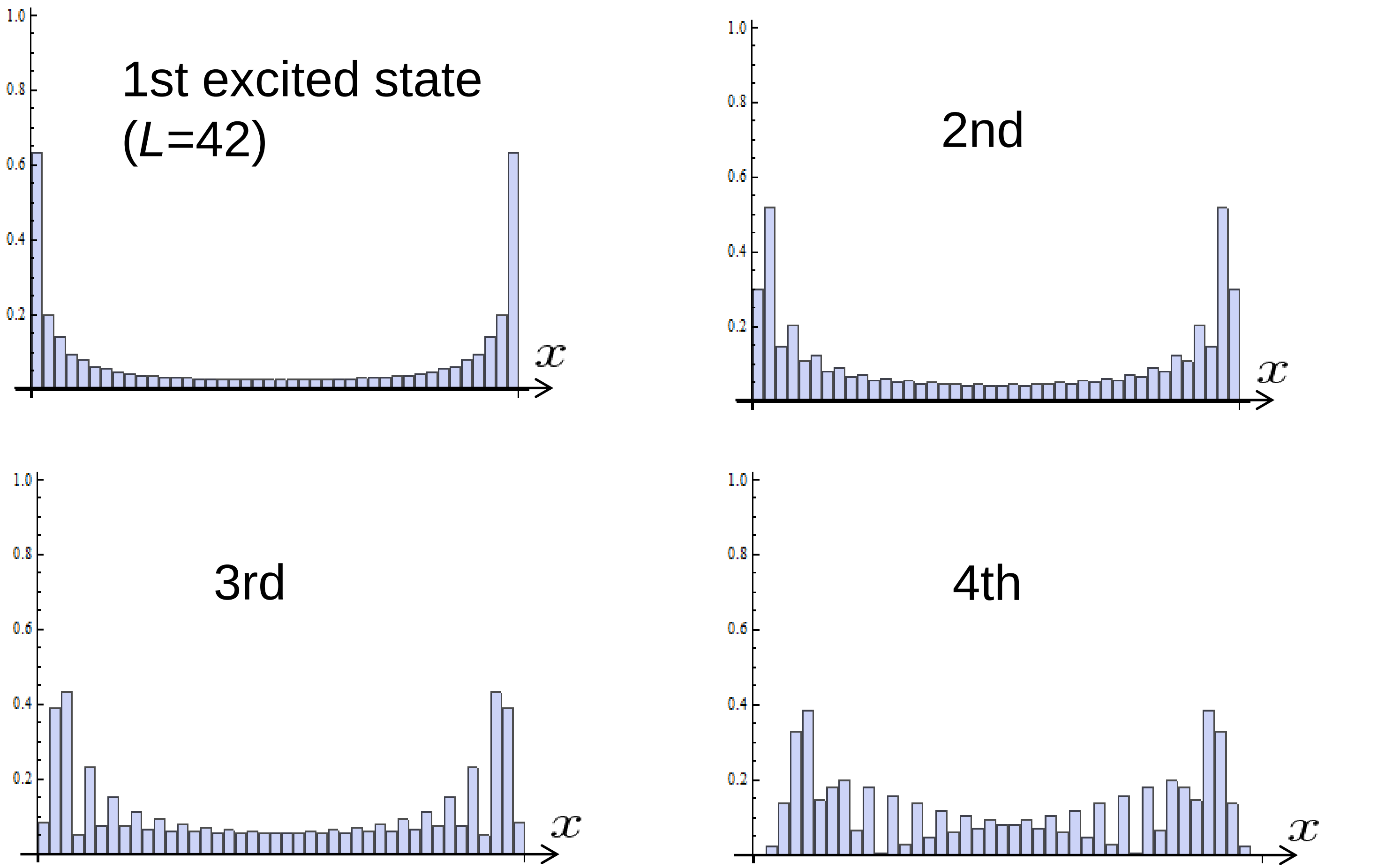


In Truong-Peschel, *IJMPB* **4** ('90), they studied a linearized model (corner Hamiltonian) and obtained

$$\epsilon_{TP}(j, L) = \frac{\pi^3}{2L(N-1)}\left(j + \frac{1}{4}\right)^2$$

Here N is the truncation number.
N~L gives the best fit to the result.
Variational estimate seems optimal.

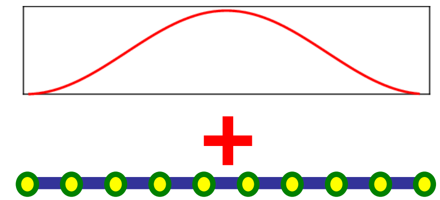
Spatial profile of 'extra' states



Low-energy states ~ edge states

Summary

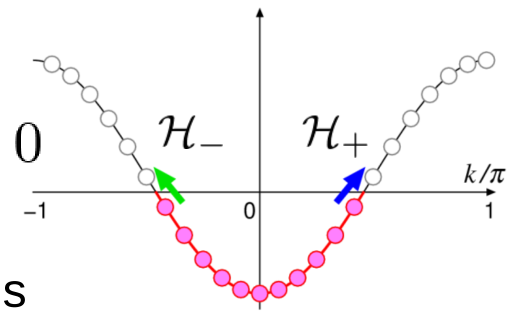
Hamiltonian with Sine-Square Deformation (SSD)
shares the same ground state with periodic chain.
→ *extremely efficient smooth boundary condition*



Mechanism of SSD

Chiral Hamiltonians annihilate the periodic g.s.

$$\mathcal{H}_{\rm SSD} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-) \qquad \mathcal{H}_\pm|0\rangle = 0$$



Ex) Free-fermion chain, anisotropic XY, Dirac fermions

CFT interpretation:

- Chiral Hamiltonians are $L_{\pm 1}$ in CFT
- *SL*(2, **C**) invariance → $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$
- The vacuum state $|0\rangle$ is a ground state of $\mathcal{H}_{\rm SSD}$.

Future directions

- Exact results for lattice SSD not reducible to free fermions
- Excited states of free fermions with SSD (Bethe ansatz?)
  Related work: SUSY QM approach (Okunishi-H.K., *JPA* **48** ('15))
- Excited states of SSD CFTs (Ishibashi-Tada, *JPA* **48** ('15), Tamura-H.K., …)