# The Entanglement Hamiltonian, SSD, and others in one-dimensional critical systems

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[Closely related works: Ishibashi-Tada (15-16), Okunishi (16)]

# Hamiltonians in CFT

- Let's start from a Hamiltonian of (1+1)d CFTs;
  On a lattice (chain), it would look like: $$H = \sum_i h_{i,i+1}$$

- Deformed evolution operator: $$H[f] = \sum_i \underline{f\left(\frac{x_i + x_{i+1}}{2}\right)} h_{i,i+1}$$
  envelope function

- E.g. Entanglement Hamiltonian: $$f(x) = \frac{R^2 - x^2}{2R}$$

- E.g. Sine-square deformation (SSD): $$f(x) = \sin^2 \frac{\pi x}{L}$$
  [Gendiar-Krcmar-Nishino (2009) ... ]

- Other applications: Rainbow chain, inhomogenous systems,
  quantum energy inequalities, etc.

# Entanglement Hamiltonians in CFT

- Reduced density matrix for a finite interval [-R, R] by tracing out its compliment

$$\rho_A = \mathrm{Tr}_{\bar{A}} |GS\rangle\langle GS|$$

- The entanglement Hamiltonian: $\rho_A \propto \exp(-H_E)$

- H_E is generically complicated.

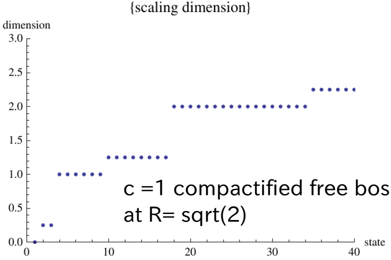
- For CFT,

$$H_E = \int dx \, \frac{(R^2 - x^2)}{2R} \mathcal{H}(x)$$

- Finite size scaling: the spectrum of (boundary) CFT: Equidistance energy levels, and 1/log(R) scaling

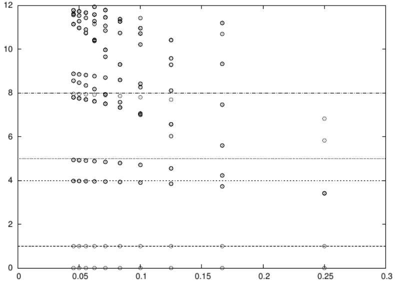
-- For a given tower of states,
all levels are equally spaced
(with degenearcy, which depends
on details of the theory)

$$H = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$$




c =1 compactified free boson
at R= sqrt(2)

$$(E - E_{\rm GS})/(E_1 - E_{\rm GS})$$




$$1/L$$

-- Level spacing scales as 1/L

# Strategy

- We will focus on critical systems (points) described by CFT.

- We will disscuss types of "deformations" $H[f] = \int dx\, f(x)\mathcal{H}$ generated by various conformal maps.

- Put differently, we are interested in deformations which we can "undo" by conformal maps.

- Will discuss spectral properties (finite size scaling) of $H[f]$

# Warm-up

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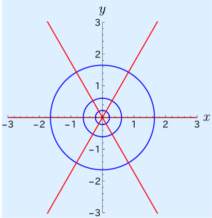
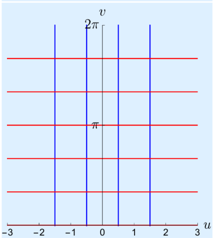
$z = x + iy$




$w(z) = \log z$

$w = u + iv$



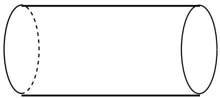

- CFT on the plane <--> CFT on a cylinder (quantum lattice model on a circle)

- Radial evolution <--> Hamiltonian (1/L scaling)
  (Dilatation)

- Angular evolution <--> Hamiltonian with boundary
  ("Rindler" or "Modular"
  or "Entanglement" Hamiltonian)

# Finite size scaling of CFT [Cardy]

- CFT on a cylinder of circumference L:



$$\tilde{H} = \frac{1}{2\pi}\int_0^L dv\,\tilde{T}_{uu}(u_0, v)$$

$$\tilde{H} = \frac{1}{2\pi}\oint_{C_w} dw\,\tilde{T}(w) + (\mathrm{anti-hol})$$

$$\tilde{T}_{uu}(w) = \tilde{T}(w) + \bar{\tilde{T}}(\bar{w})$$

- Conformal map: cylinder --> plane $\quad w = \frac{L}{2\pi}\log z.$

$$\tilde{T}(w) = \left(\frac{2\pi}{L}\right)^2\left[z^2 T(z) - \frac{c}{24}\right]$$

$$\tilde{H} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{24}\right)$$

$$\begin{aligned}&\oint_{C_w} dw\,\tilde{T}(w)\\ &= \oint_{C_z} dz\,\frac{dw}{dz}\left(\frac{2\pi}{L}\right)^2\left[z^2 T(z) - \frac{c}{24}\right]\\ &= \oint_{C_z} dz\left(\frac{L}{2\pi}\right)\left[z\,T(z) - \frac{c}{24}\frac{1}{z}\right].\end{aligned}$$

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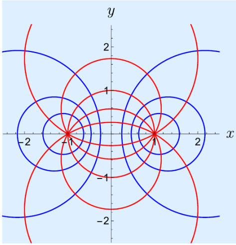
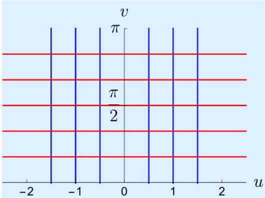
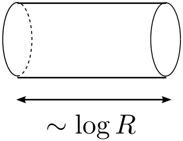



$$w(z) = \ln \frac{(z+R)}{(z-R)}$$




- Entanglement Hamiltonian on finite interval [-R, R]
  <--> Hamiltonian with boundaries

- Transforming from strip to plane:

$$H = \int du\, T_{vv}(u, v_0 = \pi) = \int_{-R}^{+R} dx\, \frac{(x-R)(x+R)}{2R} T_{yy}(x, y=0)$$

- Entanglement spec: 1/Log(R) scaling.

[See, e.g: Casini-Huerta-Myers (11), Cardy @ 2015 KITP conference]

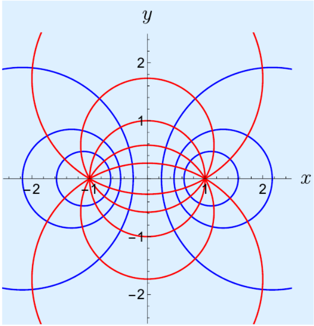
## What is the evolution orthogonal to the evolution by Entanglement H ?

- Let's take a circle as a "space"  (Cauchy surface)

$$\left(x + \frac{\cosh u_0}{\sinh u_0} R\right)^2 + y^2 = \frac{R^2}{(\sinh u_0)^2}$$

We have chosen: $r_0 := \dfrac{R}{\sinh u_0}$

Circumference: $L = 2\pi r_0$




- Evolution operator

$$H = \int_0^\pi dv\, T_{uu}(u_0, v) = r_0^2 \int_0^{2\pi} d\theta\, \frac{\cos\theta + \cosh u_0}{\sinh u_0} T_{rr}(r, \theta)$$

- "Regularized" version of the SSD:

$$H = \frac{L}{2\pi} \frac{1}{\sinh u_0} \int_0^L ds \left(\cos\frac{2\pi s}{L} + \cosh u_0\right) T_{rr}\left(r = \frac{L}{2\pi}, \theta = \frac{2\pi s}{L}\right)$$

# Summary

- The following Hamiltonian on a cylinder $(w = u_0 + iv)$

$$H = \frac{1}{2\pi}\int_0^L d\sigma \left[p - \frac{q}{2}\left(e^{\frac{-2\pi\xi}{L}} + e^{\frac{2\pi\xi}{L}}\right)\right] T_{\tau\tau}(\tau_0, \sigma)$$

$$p = r\cosh\lambda$$
$$q = r\sinh\lambda$$

can be mapped into the standard Hamiltonian on a cyinder by $(\xi = \tau_0 + i\sigma)$:

$$\xi = \frac{L}{2\pi}\log\left[\frac{\cosh\frac{\pi w}{L} + \cosh\lambda}{\sinh\lambda}\right]$$

$$H = \frac{r}{2\pi}\int_0^L dv\, T_{uu}(u_0, v)$$

# "Regularized" SSD

$$H = \frac{L}{2\pi}\frac{1}{\sinh u_0}\int_0^L ds \left(\cos\frac{2\pi s}{L} + \cosh u_0\right) T_{rr}\left(r = \frac{L}{2\pi}, \theta = \frac{2\pi s}{L}\right)$$

- By construction, this operator has the spectrum of CFT on a circle with level spacing of order one.

- Define: $$H_{\rm rSSD} = \int_0^L ds \left(\cos\frac{2\pi s}{L} + \cosh u_0\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right)$$

- The envelope function:

$$\begin{aligned} f(s) &= \cos\left(\frac{2\pi s}{L}\right) + \cosh u_0 \\ &= \cos\left(\frac{2\pi s}{L}\right) + \sqrt{1 + \left(\frac{2\pi R}{L}\right)^2}. \end{aligned}$$

$$\begin{aligned} f(s) &\overset{R\to 0}{\to} \cos\left(\frac{2\pi s}{L}\right) + 1 \\ &= \cos^2\left(\frac{\pi s}{L}\right) = \sin^2\left[\frac{\pi}{L}\left(s - \frac{L}{2}\right)\right]. \end{aligned}$$

- "Regularized" version of the SSD:
  R, the distance between vortices, is the regularziation parameter.

- Scalilng:
  (i) fix u0, change R --> 1/L scaling

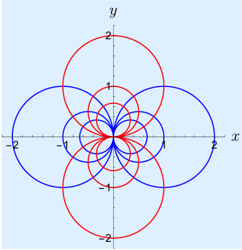
$$\sim \frac{\sinh u_0}{L} = \frac{1}{2\pi}\frac{(\sinh u_0)^2}{R} \sim \frac{1}{R}.$$

  (ii) fix R, change u_0 --> 1/L^2 scaling

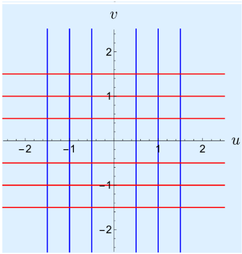
$$\sim \frac{\sinh u_0}{L} = \frac{1}{2\pi}\frac{(\sinh u_0)^2}{R} \sim \frac{1}{L^2}.$$

# The dipolar limit

- Can take the dipolar limit $R \to 0$ rSSD --> SSD:




$$w(z) = 1/z$$



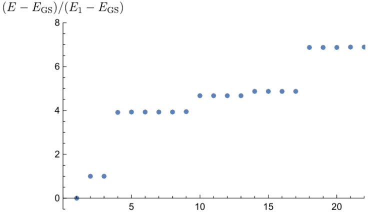

- In the dipolar limit, the w-plane (u-v plane) is an infinit plane
  --> Infinite system length limit, continuum spectrum [Ishibashi-Tada (15,16)]

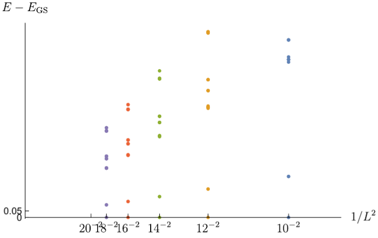
$$\begin{aligned} H &= \int_{-\infty}^{+\infty} dv\, T_{uu}(u_0, v) = 4r_0^3 \int_0^{2\pi} d\phi\, \sin^2(\phi/2) T_{rr}(r_0, \theta) \\ &= \frac{L^2}{\pi^2} \int_0^L ds\, \sin^2\left(\frac{\pi s}{L}\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right) \end{aligned}$$

- The prefactor L^2 is indicative of the 1/L^2 scaling seen in numerics.

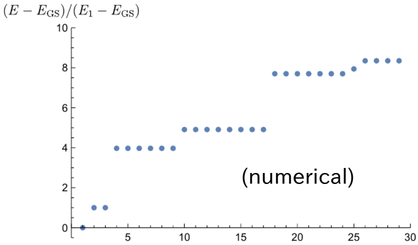
# Numerics (rSSD)

$$H = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$$




Regularized SSD spectrum

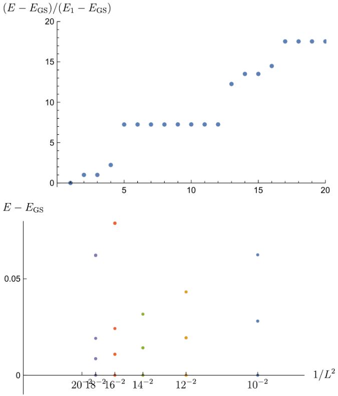



Physical spectrum (PBC)

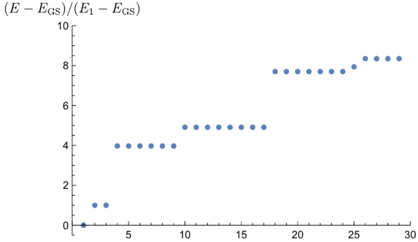
$$H = \sum_i f(x_i + 1/2) h_{i,i+1}$$

$$f(x) = \cos \frac{2\pi x}{L} + \sqrt{1 + \left(\frac{2\pi R}{L}\right)^2}$$

# Numerics (SSD)



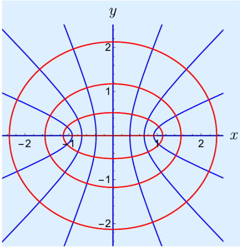

SSD



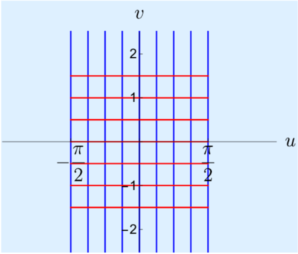

Physical spectrum (PBC)

SSD spectrum does not much physical spectrum,
1/L^2 scaling

# Other examples

- 


$$z = \sin(w)$$




infinite stripe

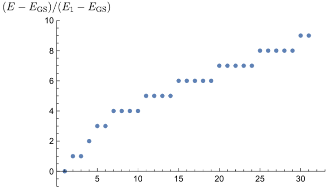
$-\pi/2 < u < \pi/2$

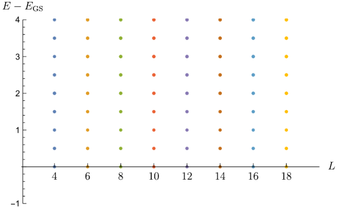
$-\infty < v < +\infty$

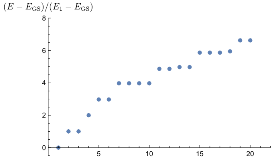
- Engineering conformal map and evolution operator

- "square root deformation" $\tilde{H} = \int_{-R+\epsilon}^{+R-\epsilon} dx \sqrt{R^2 - x^2}\, T_{yy}$

- Known in the context of "perfect state transfer"
  (Thanks: Hosho Katsura)




Physical spectrum (OBC)

"Square root" deformation

# Summary

- Setup a general discusion of "deformed" Hamiltonians in CFTs

- Proposed a "regularized" version of SSD (rSSD).

- Original SSD can be viewed as a "singular" limit of rSSD

- Spectrum of rSSD is easy to understand.
  Shed light on 1/L^2 scaling of SSD.