Sine Square Deformation And Related Topics 2019

iTHEMS workshop held at RIKEN Wako Campus on July 11, 2019

On July 11th, 2019, the workshop on "Sine-Square Deformation and related topics 2019," was held at RIKEN Wako Campus. It is a sequel to the previously held workshop on June 22nd, 2017 at RIKEN Wako Campus. The aim of the workshop is to study Sine-square deformation (SSD), which is a new type of boundary condition at which the coupling constant of the system is spatially modulated. Since its inception, SSD has been studied in various contexts including string theory, condensed matter physics and quantum field theory. This workshop provided a unique opportunity for the researchers working on the subject from various disciplines to get together and exchange their knowledge and perspectives. We were able to invite a large portion of the researchers who are actively contributing to the subject, including Dr. Nishino, who initiated the subject.
This document contains the slide shown by the speakers at the workshop. Since all the presentations were given in Japanese, some part of the slides are in Japanese

Organizers:
Nobuyuki Ishibashi (Tsukuba), Hosho Katsura (Tokyo),
Kouichi Okunishi (Niigata), Tsukasa Tada(RIKEN)

Workshop Program

10:00 ~ 10:45　Tomotoshi Nishino (Kobe) 「エネルギースケール変換から SSD へ、辿った経緯と未着手の問題」

10:55 ~ 11:25　Hosho Katsura (Tokyo) 「ディリクレ・ノイマン混合境界条件と half SSD」

11:35 ~ 12:20　Tomohiko Takahashi (Nara W.) 「タキオン真空とサイン二乗変形」

lunch

13:20 ~ 14:05　Shinsei Ryu (Chicago), "Holographic duals of inhomogeneous 1d quantum many-body systems”

14:15 ~ 14:45　Toshiya Hikihara (Gunma) 「様々なエネルギースケール変換による一次元量子系の特性変化」

14:45 ~ 15:00　Syoji Zeze (Yokote Seiryou H.), ”Virasoro algebra in K-space”

break

15:30 ~ 16:15　Naokazu Shibata (Tohoku) 「SSDを用いたフラストレート量子スピン系および多自由度電子系の解析」

16:25 ~ 17:10　Kouichi Okunishi (Niigata) 「XXZ鎖における格子Unruh効果と世界線エンタングルメント」

17:20 ~ 18:00　Tsukasa Tada(RIKEN), ”Perspectives from Sine-square deformation on conformal field theories”

Quantum-Classical correspondence and Energy Scale Deformations

Tomotoshi Nishino (Kobe Univ.)
Roman Krcmar (SAS)
Andrej Gendiar (SAS)
… anonymous referee …

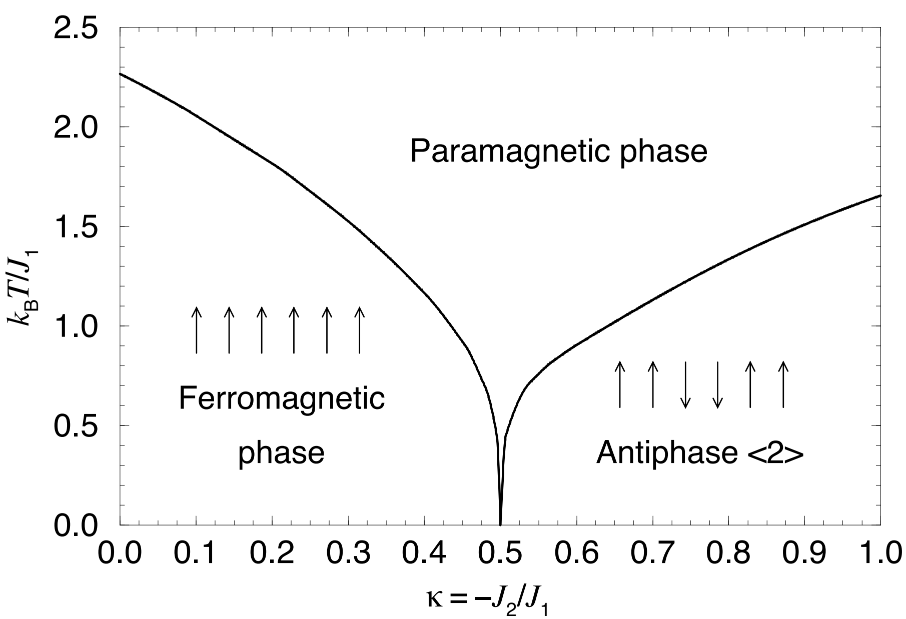
arXiv:0810.0622

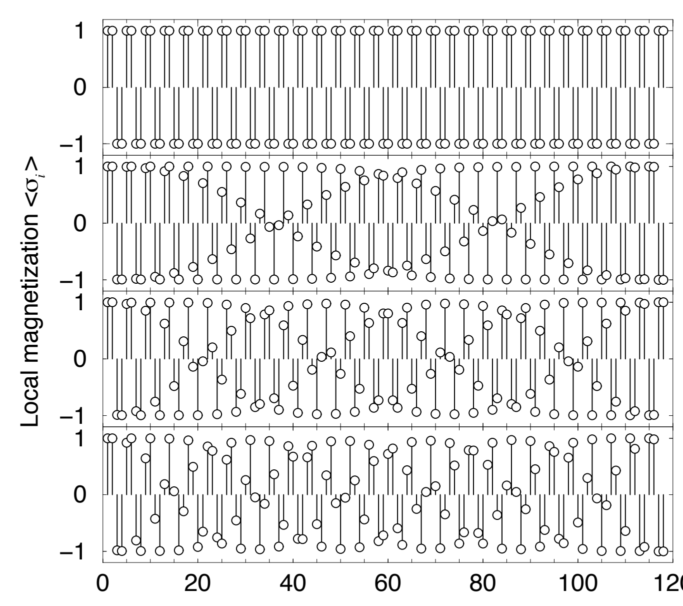
*** Uniform Hamiltonian does not always have uniform ground state.**

- Charge/Spin density wave, commensurate or incommensurate

- ex. Axial Next Nearest Neighbor Ising (ANNNI) model

$$\mathcal{H} = -J_1 \sum_{i,j} \sigma_{i,j}(\sigma_{i+1,j} + \sigma_{i,j+1}) - J_2 \sum_{i,j} \sigma_{i,j}\sigma_{i+2,j}$$





Energy Scale Deformation

*** There is a modulated Hamiltonian whose ground state is uniform.**

- empty state of any Fermionic system (too trivial!)
- (modulated/inhomogeneous) AKLT Hamiltonian

  since H = sum of projectors, and pre factor can be arbitral

- Slow energy scale modulation would not affect a gapped ground state

  if the modulation is slow enough (or gap is wide enough)

- Exponential Deformation (Wilson, …, Okunishi)

wilson lattice arXiv:1001.2594

$$\mathcal{H}_\lambda = \sum_{n=1}^{N-1} e^{\lambda n} (c^\dagger_{n+1} c_n + c^\dagger_n c_{n+1})$$

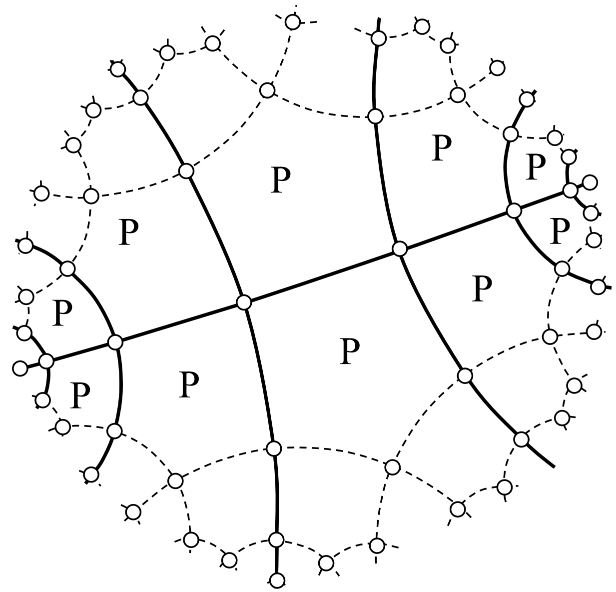
general framework arXiv:cond-mat/0702581

$$H_N(\Lambda) = \sum_{n=1}^{N-1} \Lambda^{N-n-1} h_{n,n+1},$$

a classical counterpart: Hyperbolic Lattice

arXiv:0704.1949

Ising model on Hyperbolic Lattice



- there is ferro-para phase transition
- always off critical
- row-to-row transfer matrix can be defined

- is it possible to find out the corresponding quantum Hamiltonian? (I have no answer)

probably, in anisotropic limit (how to define this limit?), one reaches the hyperbolic deformation. arXiv:0808.3858

$$
\begin{aligned}
H^{\cosh}(\lambda) &= \frac{1}{2}\left[H^{\exp}(\lambda) + H^{\exp}(-\lambda)\right] \\
&= \sum_{j=-N}^{N} \cosh j\lambda \; h_{j,j+1} \,.
\end{aligned}
$$

ground-state is uniform, except for the edge state, as it was observed in the case of exp. deformation.

a path to "spherical" deformation

* Corner Hamiltonian ~ Entanglement Hamiltonian

- Okunishi proposed a quantum counterpart of CTMRG

$$K_{\mathrm{N}} = \sum_{n=1}^{N-1} n h_{n,n+1},$$

cond-mat/0507195

- Hyperbolic "deformation" can be considered

$$H^{\sinh}(\lambda) = \sum_{j=-N}^{N} \sinh j\lambda \; h_{j,j+1} \,,$$

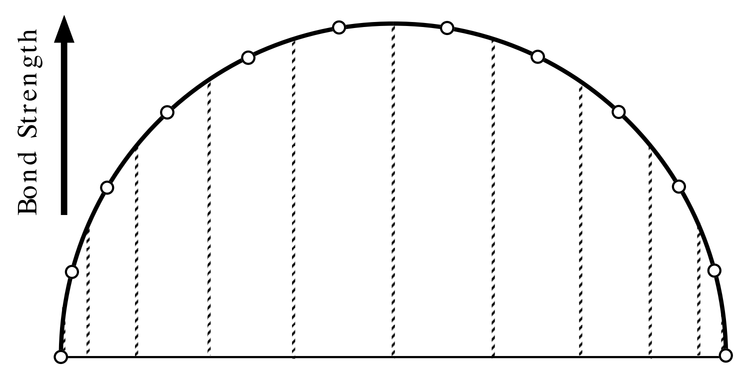
arXiv:0808.3858

* History in physics suggests the generalization to trigonometric deformations

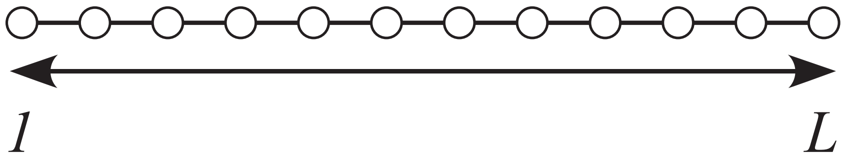
$$H_{\mathrm{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell)\, h_{\ell,\ell+1}$$

arXiv:0810.0622

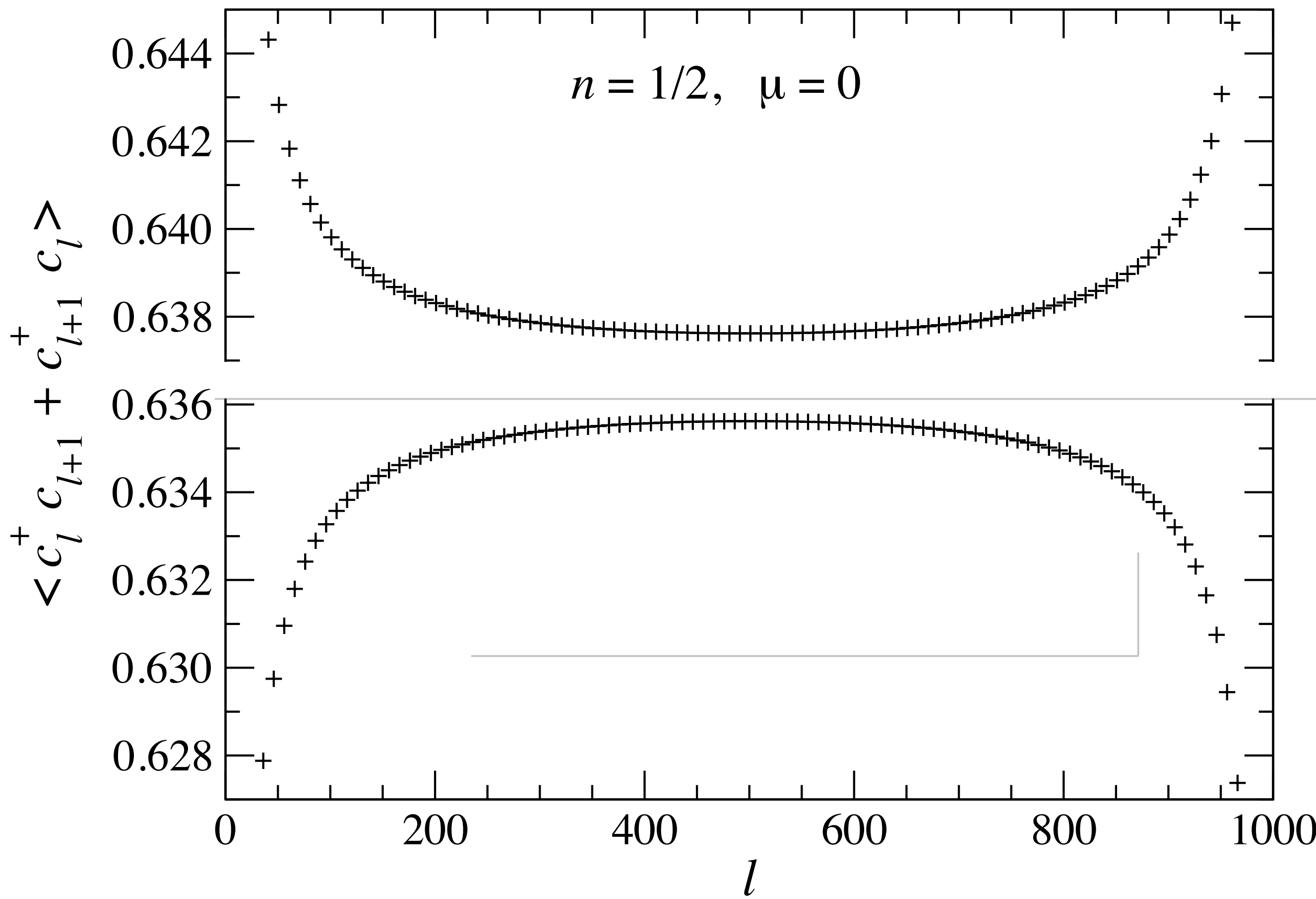
… well, the prototype was "cosine deformation", and not squared.
How can one use the deformation? (I don't know.)



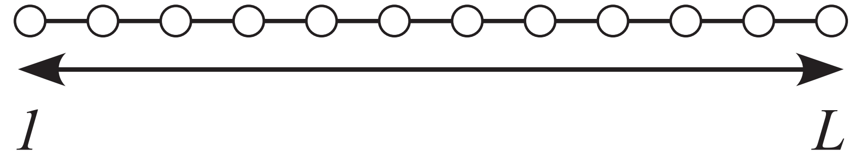
arXiv:0810.0622



最近接格子点間の「相関関数」を求めてみる。N=1000 サイトの系での計算結果は？「境界効果」であるフリーデル振動が、内部まで浸透していることがわかる。(金属表面で電子密度が振動するのも同じようなもの)



Smooth Boundary Condition



飛び移り振幅 -t を、系の両端で小さくすれば、上手く「ターミネート」できるのではないか？

PHYSICAL REVIEW LETTERS

VOLUME 71 | 27 DECEMBER 1993 | NUMBER 26

Smooth Boundary Conditions for Quantum Lattice Systems

M. Vekić and S. R. White

Department of Physics, University of California, Irvine, California 92717

(Received 1 September 1993)

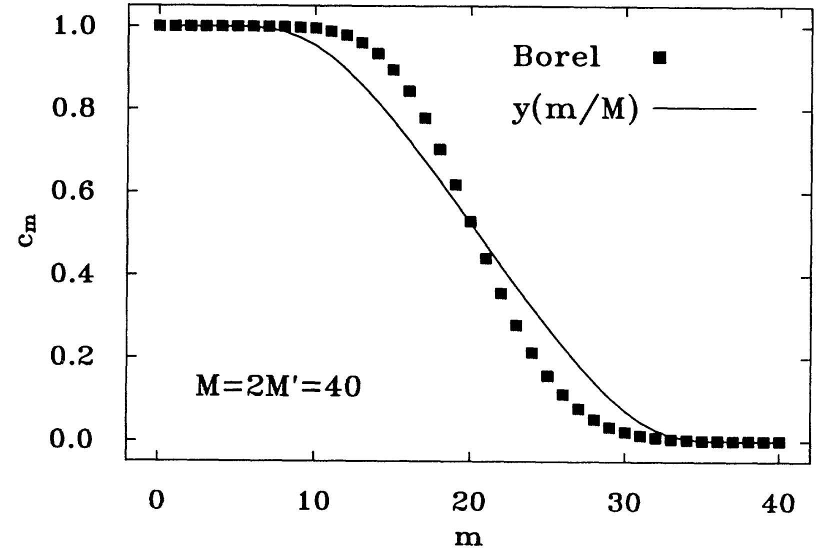
We introduce a new type of boundary conditions, *smooth boundary conditions*, for numerical studies of quantum lattice systems. In a number of circumstances, these boundary conditions have substantially smaller finite-size effects than periodic or open boundary conditions. They can be applied to nearly any short-ranged Hamiltonian system in any dimensionality and within almost any type of numerical approach.

PACS numbers: 02.70.−c, 05.30.Fk, 75.10.Jm

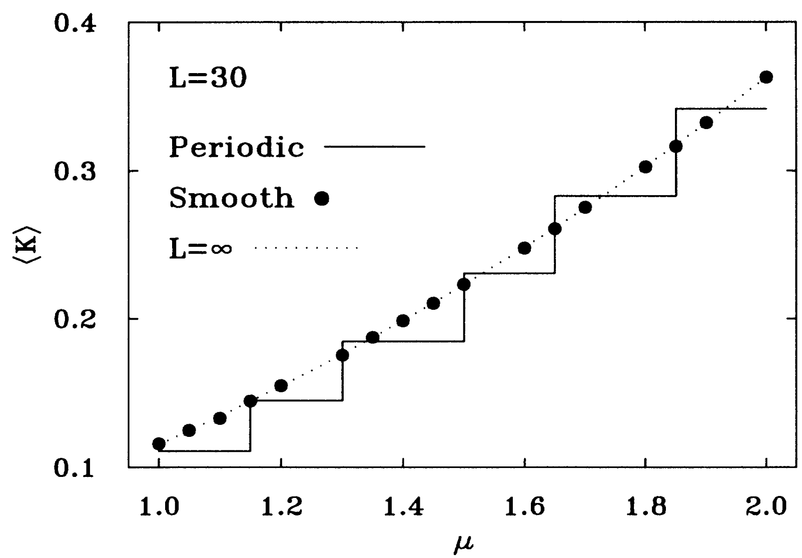
まあまあ上手く行く

White の成果

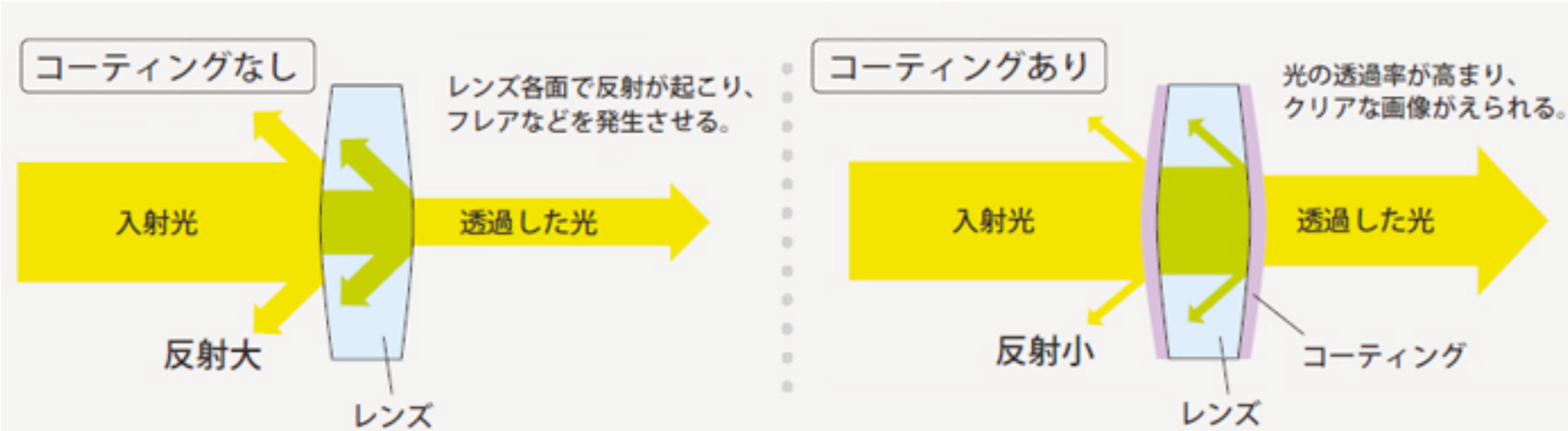
飛び移り振幅 -t の、系の両端でのスムージング関数



化学ポテンシャル変化に対する粒子密度の変化



レンズのコーティングもまた同じ



← 望遠鏡の善し悪しは、対物レンズのコーティングを見ると、おおよそ推測できることが多い。
（粗悪品は値段の割に口径が大！）

a path to "spherical" deformation

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- Okunishi proposed a quantum counterpart of CTMRG

$$K_{\mathrm{N}} = \sum_{n=1}^{N-1} n h_{n,n+1},$$

cond-mat/0507195

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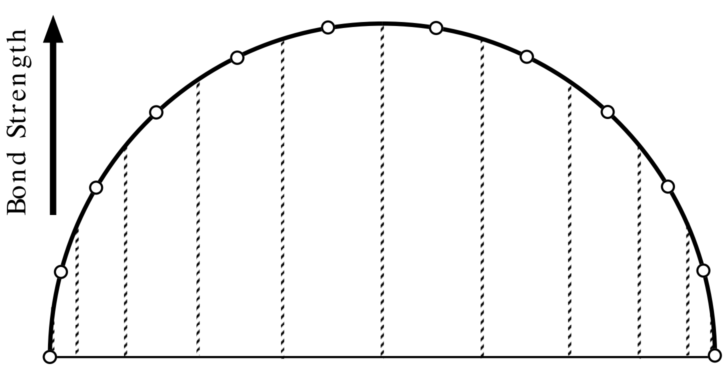


arXiv:0810.0622

arXiv.org > cond-mat > arXiv:0810.0622

Condensed Matter > Strongly Correlated Electrons

Spherical Deformation for One-dimensional Quant

Andrej Gendiar, Roman Krcmar, Tomotoshi Nishino

(Submitted on 3 Oct 2008 (v1), last revised 27 Dec 2010 (this version, v6))

Submission history

From: Andrej Gendiar [view email]

[v1] Fri, 3 Oct 2008 12:09:55 UTC (58 KB)
[v2] Mon, 30 Mar 2009 14:55:38 UTC (71 KB)
[v3] Fri, 19 Jun 2009 14:47:53 UTC (308 KB)
[v4] Tue, 14 Jul 2009 17:15:13 UTC (326 KB)
[v5] Thu, 16 Jul 2009 16:57:13 UTC (326 KB)
[v6] Mon, 27 Dec 2010 08:05:20 UTC (447 KB)

[v1] $$H_{\rm S}^{N} = -t \sum_{\ell=-N/2}^{N/2-2} \cos\left(\frac{\ell+1}{N-1}\pi\right) \left(c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell\right)$$

… finally we reach sin^2 form, … almost ACCIDENTALLY

Errara published in Prog. Theor. Phys. **123** (2010), 393. 393

Errata

Spherical Deformation for one-dimensional Quantum Systems

Andrej Gendiar, Roman Krcmar, and Tomotoshi Nishino

Prog. Theor. Phys. **122** (2009), 953.

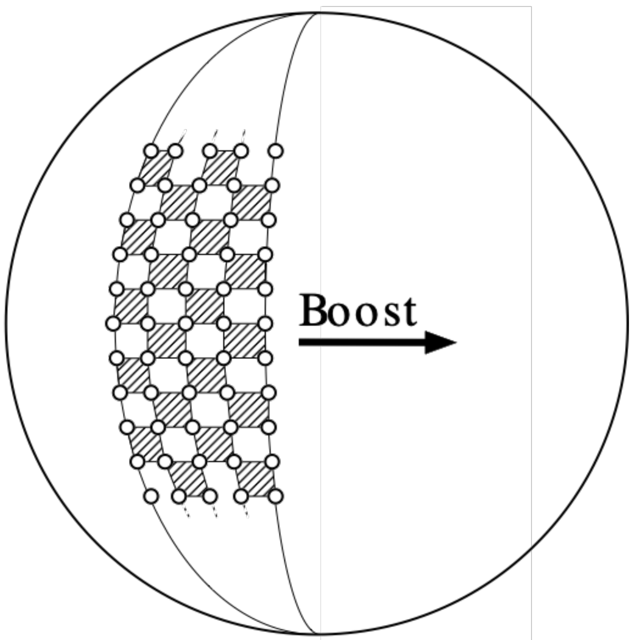
In the article we have published, we studied the finite-size correction to the energy per site E^N/N for the spherically deformed free fermion lattice, whose Hamiltonian is given by

$$\hat{H}_{\rm S}^{(n)} = \sum_{\ell=1}^{N-1} \left[\sin\frac{\ell\pi}{N}\right]^n \left(-t\,\hat{c}_\ell^\dagger \hat{c}_{\ell+1} - t\,\hat{c}_{\ell+1}^\dagger \hat{c}_\ell - \mu\frac{\hat{c}_\ell^\dagger \hat{c}_\ell + \hat{c}_{\ell+1}^\dagger \hat{c}_{\ell+1}}{2}\right) \tag{1}$$

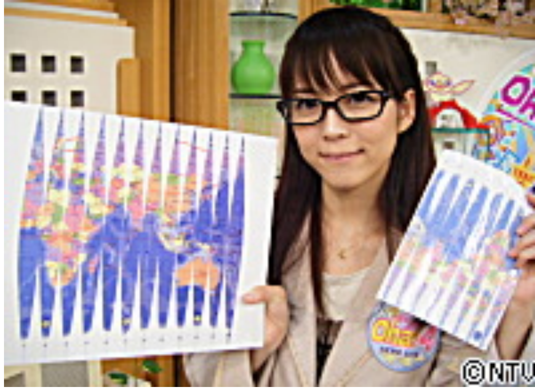
What happened?

- I visited Aachen, to discuss with Andrej Gendiar in 2008.

… we considered a way of reducing the boundary effect in 1D chain.



The following picture came up, though I do not understand what it is even now. (open problem) a sphere has no border



let us focus on the width of each piece of paper.

$$\mathcal{H}^{(N)}_{\rm sine} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N}\right)\right]^m (c^\dagger_j c_{j+1} + c^\dagger_{j+1} c_j)$$

Major contribution came from Andrej Gendiar

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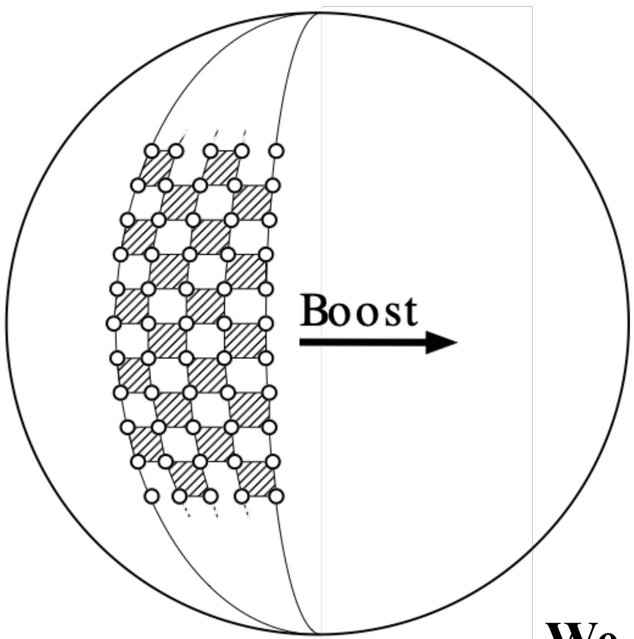

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any way, we checked the “cosine deformation” on the free fermion lattice, and confirmed that it reduces the boundary effect.

$$H_{\mathrm{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell)\, h_{\ell,\ell+1}$$

We report the result as [v1] of arXiv:0810.0622

ATTENTION: we submit [v1] to Prog. Theor. Phys.
Referee pointed that the boundary effect is reduced, but still there is.

- Andrej proposed to consider cos^n also, since the function falls to 0 MORE SMOOTHLY than cos^1.

- I denied Andrej’s proposal, since cos^n contradict the above SPHERE.

What happened?

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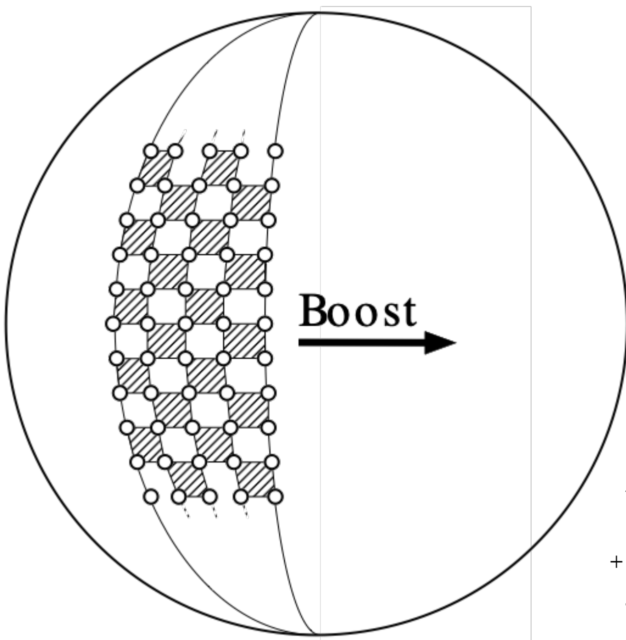
… we considered a way of reducing the boundary effect in 1D chain.



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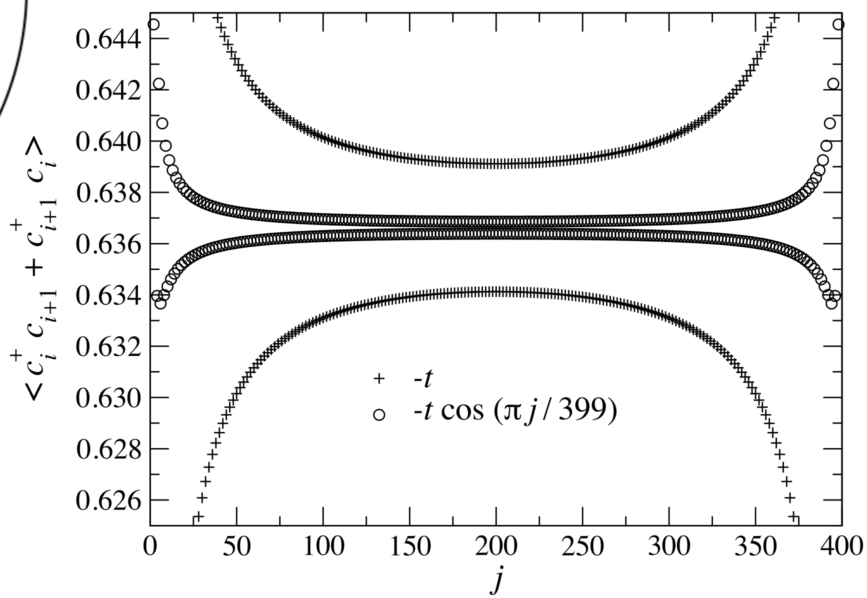


Figure 2. Expectation value $\langle c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell \rangle$ of the spherically deformed lattice Fermion model when $N = 400$. For comparison, we also plot the same expectation value for the undeformed case.

What happened? - Andrej was right, and there is one another side story.

[ERRATA] arXiv:0810.0622

In the article we have published, we studied the finite-size correction to the energy per site E^N/N for the spherically deformed free fermion lattice, whose Hamiltonian is given by

$$\hat{H}_{\mathrm{S}}^{(n)} = \sum_{\ell=1}^{N-1} \left[\sin \frac{\ell\pi}{N} \right]^n \left(-t\, \hat{c}_\ell^\dagger \hat{c}_{\ell+1} - t\, \hat{c}_{\ell+1}^\dagger \hat{c}_\ell - \mu \frac{\hat{c}_\ell^\dagger \hat{c}_\ell + \hat{c}_{\ell+1}^\dagger \hat{c}_{\ell+1}}{2} \right) \tag{1}$$

for the case $n = 1$. While we proceeded to a further study on the spherical deformation, we noticed the data shown in Figs. 2-7 were incorrect, and these figures corresponded to the Hamiltonian for the case $n = 2$. This error happened due to a very primitive confusion in the file name of computational source codes, and we misused the data with $n = 2$, instead of $n = 1$. We show appropriate data for the typical case $\mu = 0$, which corresponds to the half filling.

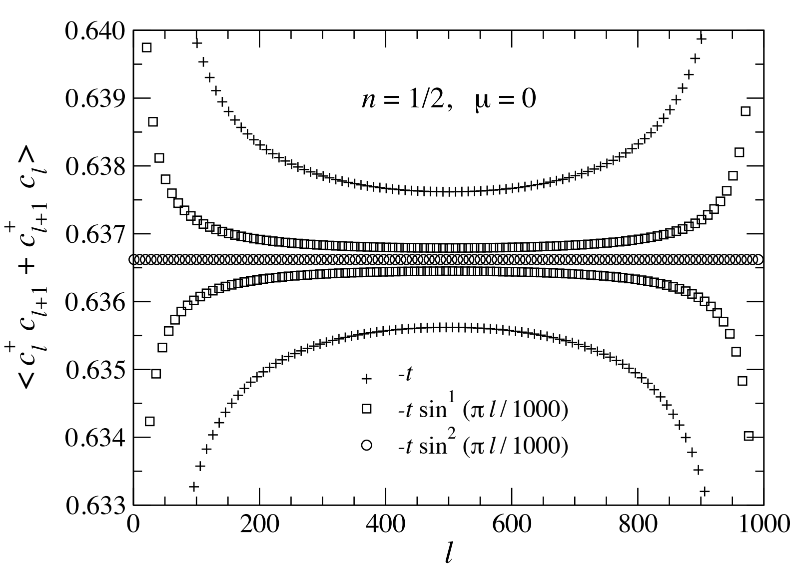


Fig. 1. Bond correlations at half filling calculated for $\hat{H}_{\mathrm{S}}^{(n)}$ with $n = 0$, 1, and 2.

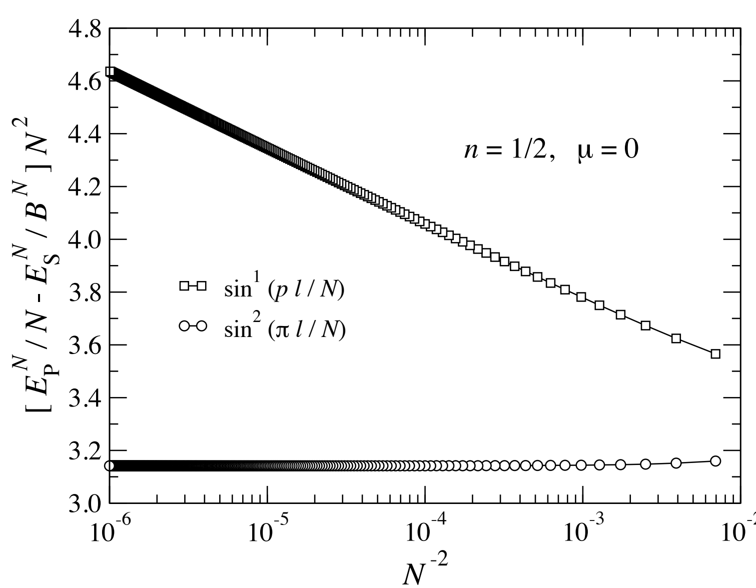


Fig. 2. Finite-size corrections to the energy.

Home Works (Conjectures)

Extension to higher dimensional system

- It is always possible to consider Hyperbolic lattice or deformation.

- Slowest modulation on N-dimensional sphere would be an extension of SSD.

Trotter decomposition

- What is the right Trotter decomposition between curved surface with constant curvature and corresponding quantum (lattice) system.

Fuzzy space

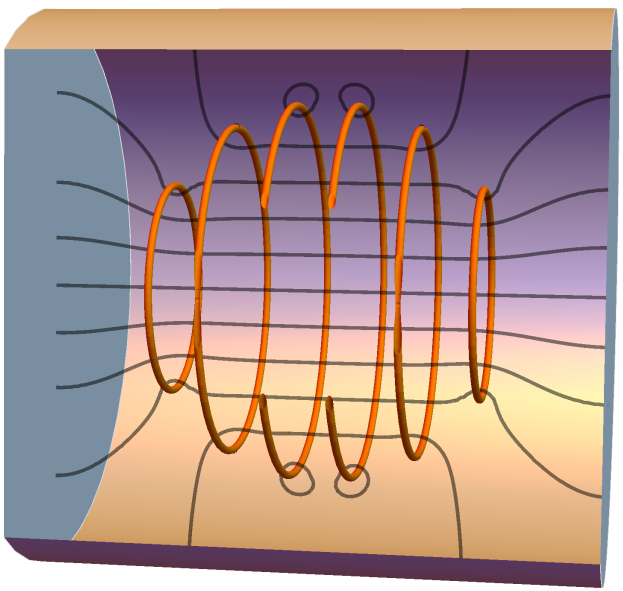
- How does non commutable space can be deformed in the manner of SSD?

[The world of Classical Physics is quite Wide]

electric magnet: should it be a cylinder?

What is the most appropriate form for the high field magnet?

Liu et al. arXiv:1907.03539



Spherical coil?

Hyperbolic helical coil?

Do find something rectangular/cylindrical

fill this space.

try to find on SNS.

Do find something rectangular/cylindrical

You are looking at
rectangular screen.

u phone, also.

以下、付録

境界条件 (Boundary Condition) というもの

同じ水面でも、その性質は容れ物に
よってエラく変化する。

a glass of water

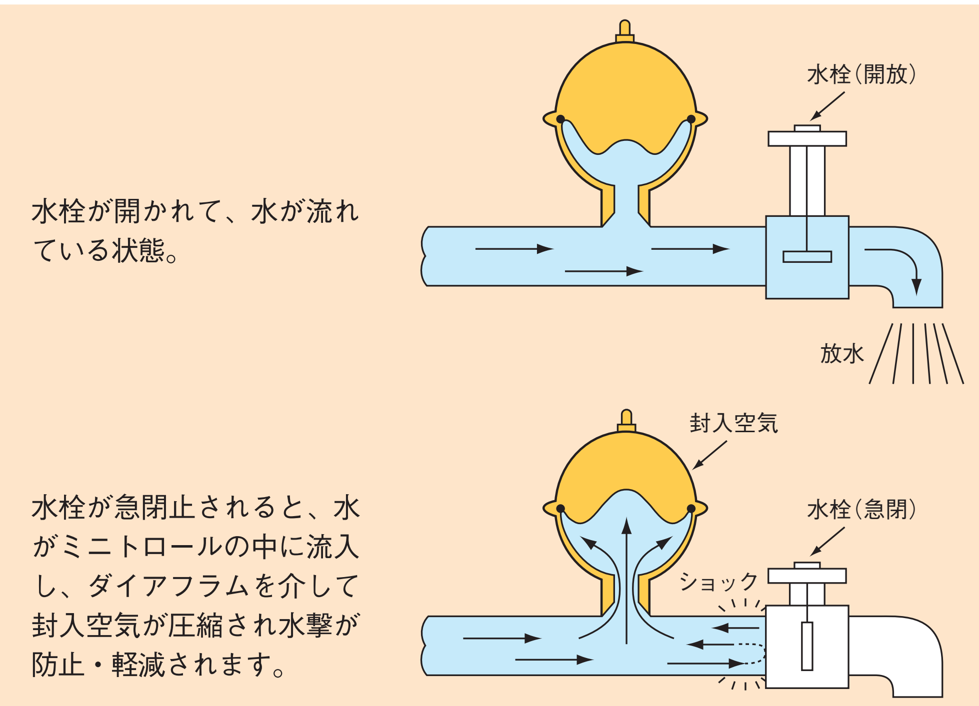
a `pacific' of water

drawing by active boundary

注) 文字が現れるのは一瞬だけ→

境界の効果は「反射」として現れる

→ 反射を減じて「無限を演出」したくなることもある

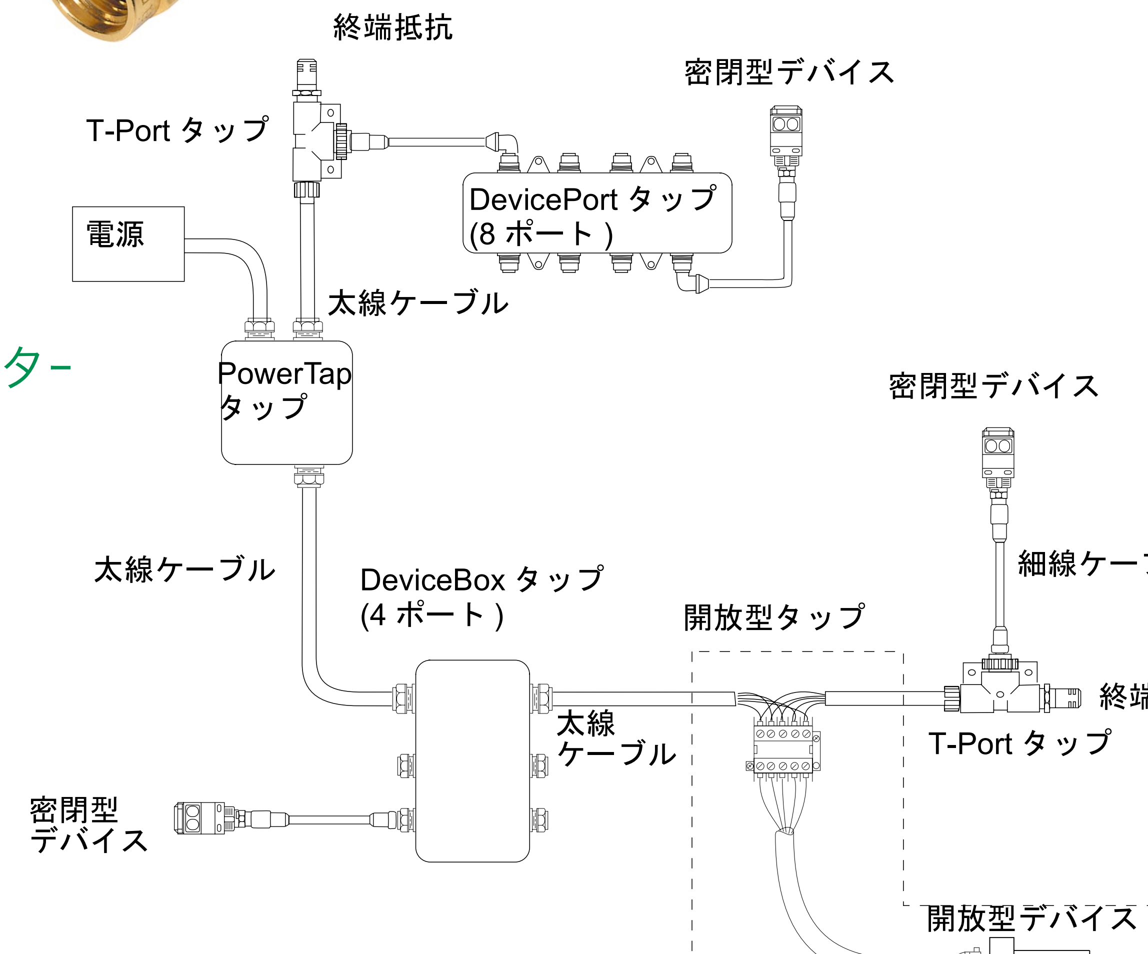


例えば水道では（通称）蛇口という境界があって、急に閉じると強い圧力波を発生させてしまう。

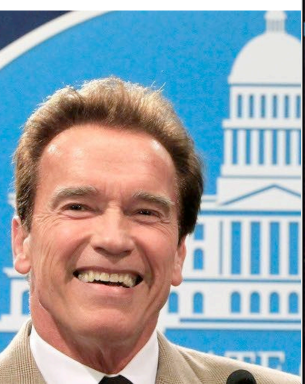
そこで、こんなものが裏で使われている、ことがある。

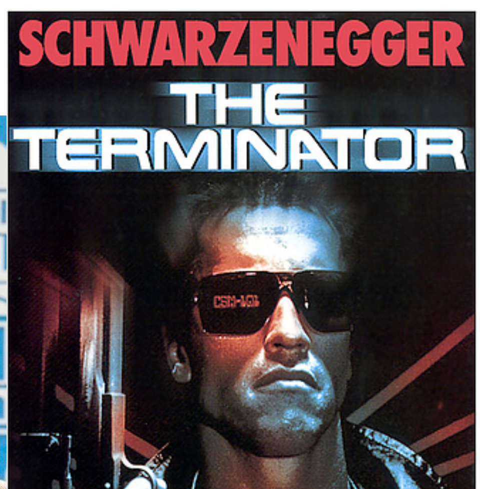
電気回路（や音響回路など）のインピーダンス整合も
境界からの信号反射を減じるための工夫である。

終端抵抗：ターミネーター
と呼ぶことが多い

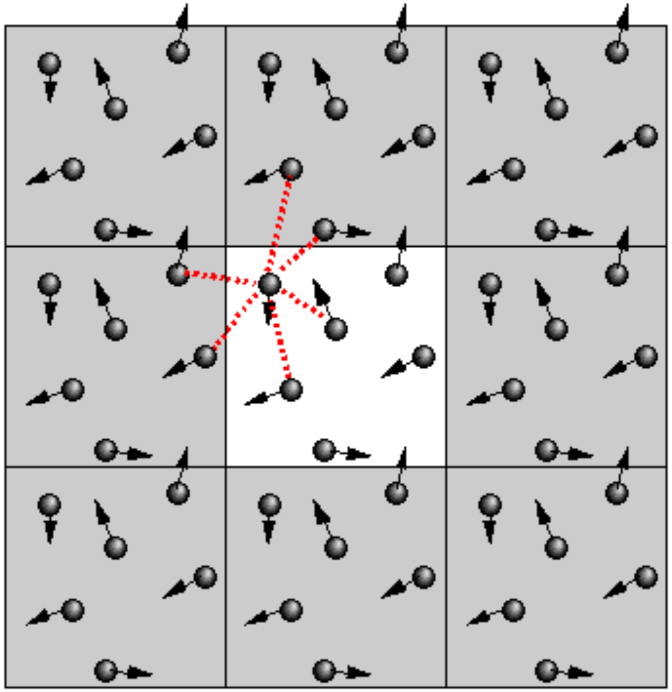


昔なつかしい SCSI のターミネーター





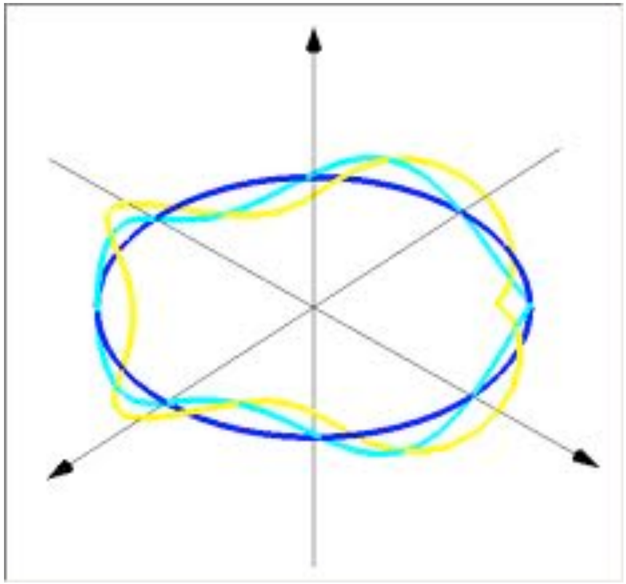
周期境界条件　境界を「てっとりばやく」消してしまう方法



むかい合う境界を「はり合わせて」しまって、境界はないけれども閉じた空間を作る。

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x) = E_n\psi_n(x)$$

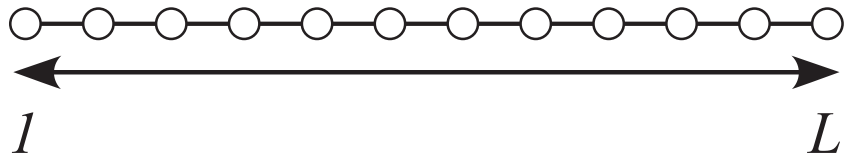
1 次元井戸型ポテンシャルの問題では、x=0 と x=L を同一視して、空間を「輪」にしてしまう。→→



... でも、ちょっと、わざとらしくない？

空間を曲げないと輪にならないよ？？

ようやく、本日の問題設定：



有限な 1 次元の格子上でフェルミ粒子が飛び移る系

$$\mathcal{H}^{(N)} = -t \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

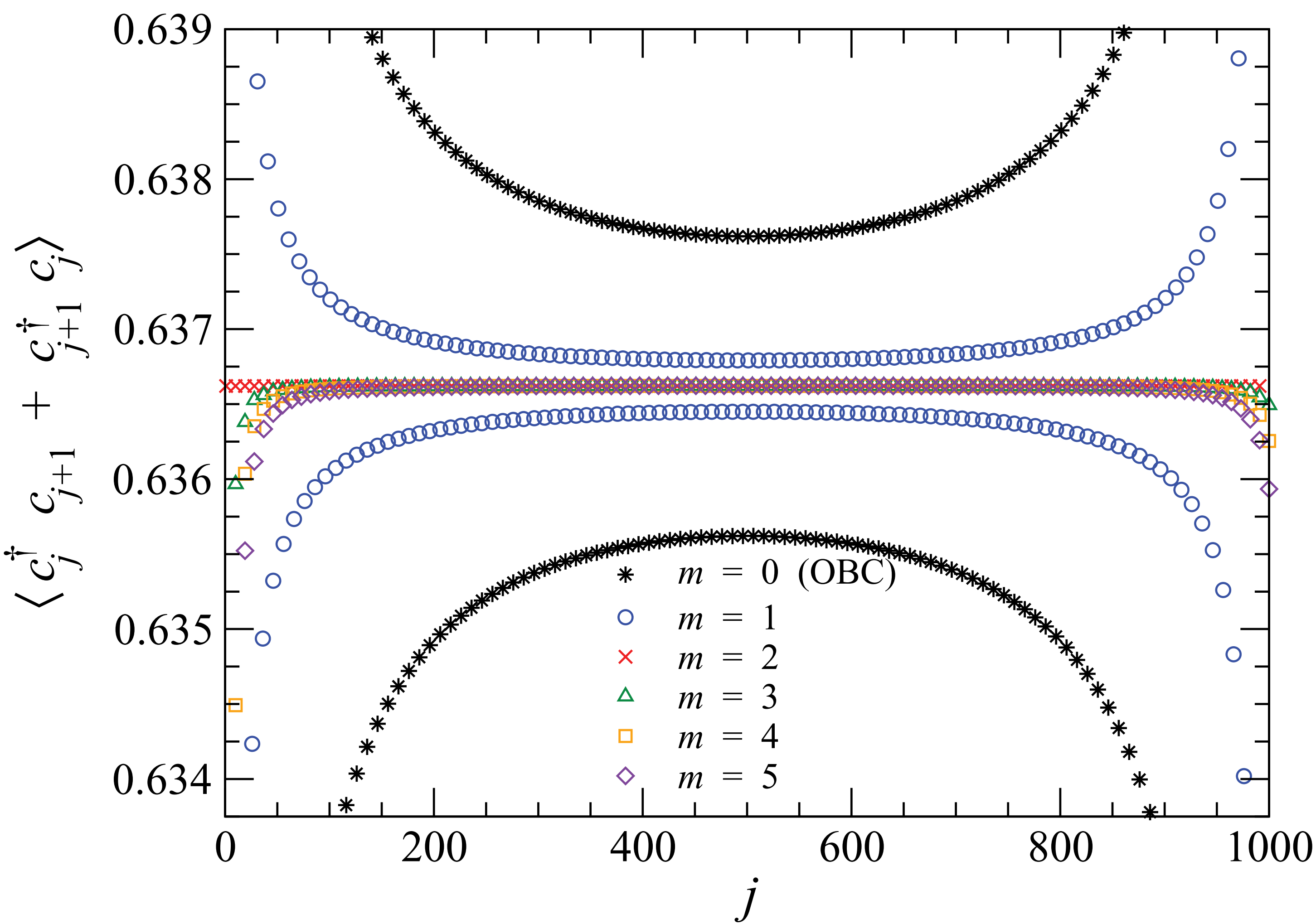
空間が離散的、つまり「並んだ格子点」で表現されている点を除いて、井戸型ポテンシャルと全く同じ問題。この格子上に置かれた粒子は、確率振幅 -t で左右の格子に飛び移り、境界である j = 1 や j = N から外へ、つまり j = 0 や j = N+1 へと出て行くことはない。

粒子を格子点の数の半分まで入れて、物理量を観察してみよう。（フェルミ粒子だから、波動関数はスレーター行列式で与えられる。）

$$\mathcal{H}_{\rm sine}^{(N)} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N}\right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

格子点の半分の数の粒子を放り込んで、再近接相関関数を計算してみると？

m = 2 で、境界効果が「ほとんど」消失してしまった。



失敗談

$$\mathcal{H}^{(N)}_{\rm sine} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N}\right)\right]^m (c^\dagger_j c_{j+1} + c^\dagger_{j+1} c_j)$$

Andrej Gendiar 氏が m = 2 の場合について集めた計算データを、メールで受け取る際の「ドサクサ」で、私は m = 1 のデータだと思い込んで論文を書いてしまった。↓そして「運悪く」そのまま掲載されてしまった .....

Progress of Theoretical Physics, Vol. 122, No. 4, October 2009

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR,[1,2] Roman KRCMAR[1] and Tomotoshi NISHINO[2,3]

科学者は「正直者」でなければならない。（但し論文を書く時「だけ」）「すんません、m=2 の間違いでした」という報告を書いて、その雑誌に掲載してもらった。

Errata

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR, Roman KRCMAR and Tomotoshi NISHINO

Prog. Theor. Phys. **122** (2009), 953.

(Received December 10, 2009; Revised December 23, 2009)

どうして境界効果が消失したの？

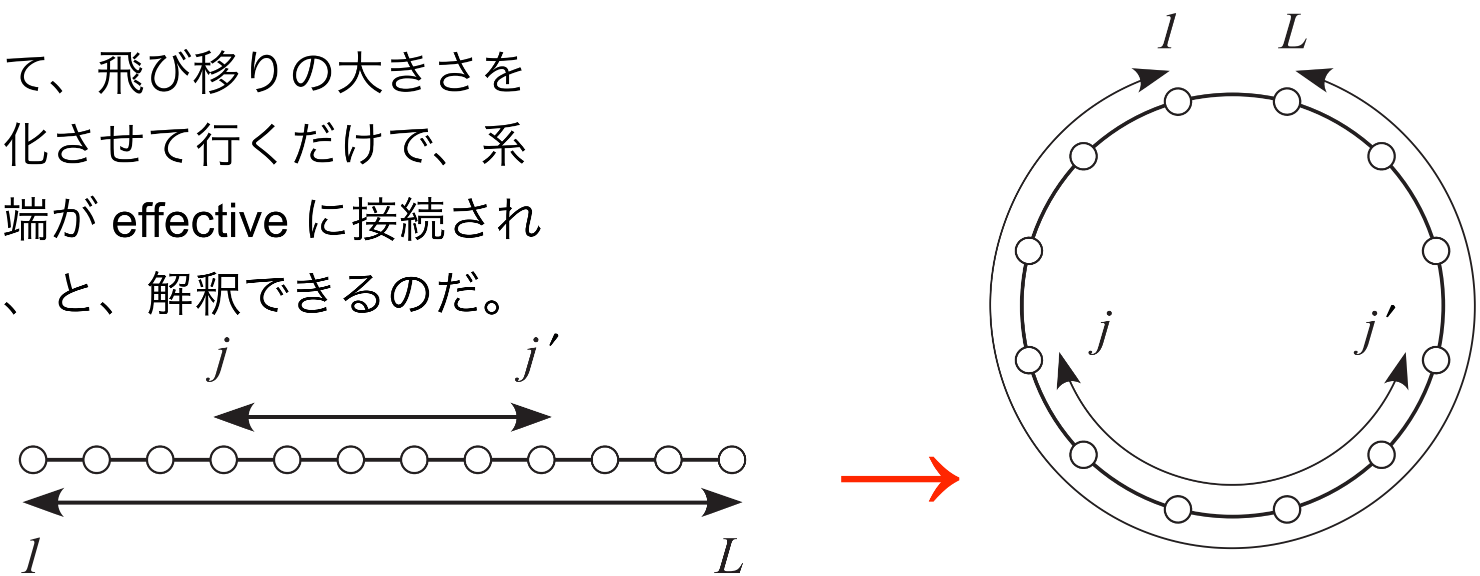
$$\mathcal{H}_{\mathrm{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N} \right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

基底波動関数を（数値的に）調べてみると、m = 2 の場合の波動関数は「hopping amplitude が一様な系で、周期境界条件を課した場合」の基底波動関数と全く同じであることがわかった。

ここが量子力学の不思議！

←広義には「波動物理学」の不思議

境界へ向けて、飛び移りの大きさを連続的に変化させて行くだけで、系の右端と左端が effective に接続されてしまった、と、解釈できるのだ。



数値計算から数理物理学へ

$$\hat{H}_S = -t \quad{}^{N-1}_{=1} \quad \sin^2 \frac{\pi}{N} \quad \hat{c}^\dagger \hat{c}_{+1} + \hat{c}^\dagger_{+1} \hat{c}$$

... 私は足し算や引き算は苦手だから、ここから
先は計算が得意な方にバトンタッチする ....

Exact ground state of the sine-square deformed XY spin chain

J. Phys. A: Math. Theor. 44 (2011) 252001

Hosho Katsura

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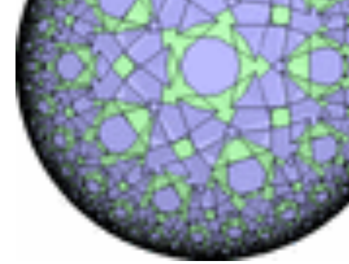
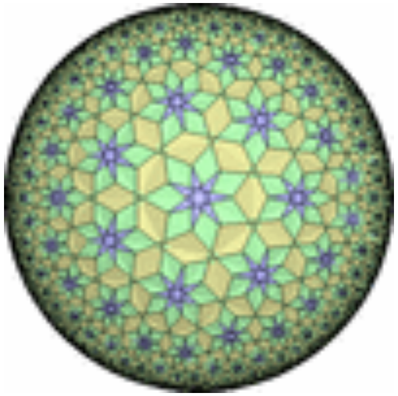
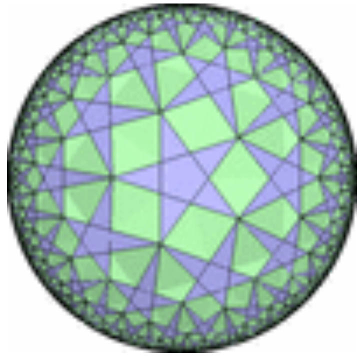
Abstract. We study the sine-square deformed quantum XY chain with open boundary conditions, in which the interaction strength at the position x in the chain of length L is proportional to the function $f_x = \sin^2[\frac{\pi}{L}(x - \frac{1}{2})]$. The model can be mapped onto a free spinless fermion model with site-dependent hopping amplitudes and on-site potentials via the Jordan-Wigner transformation. Although the single-particle eigenstates of this system cannot be obtained in closed form, it is shown that the many-body ground state is identical to that of the uniform XY chain with periodic boundary conditions. This proves a conjecture of Hikihara and Nishino [Hikihara T and Nishino T 2011 *Phys. Rev. B* **83** 060414(R)] based on numerical evidence.

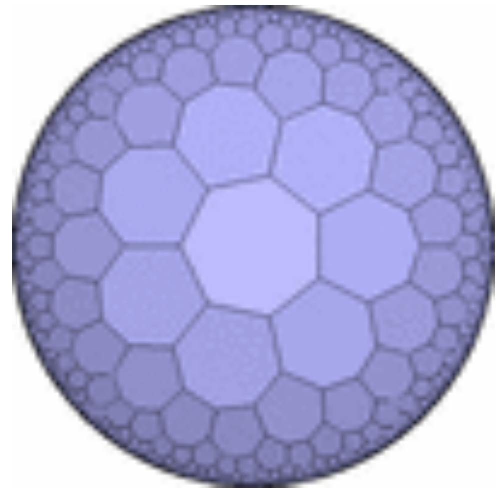
証明されちゃった!! 但し、背景の「代数」は未解決なまま ...

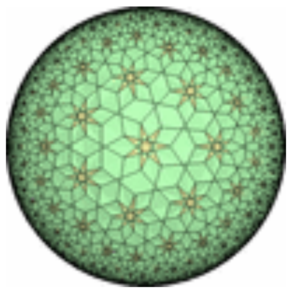
柳の下にはドジョウが百匹：あらゆる等質空間へ!!!

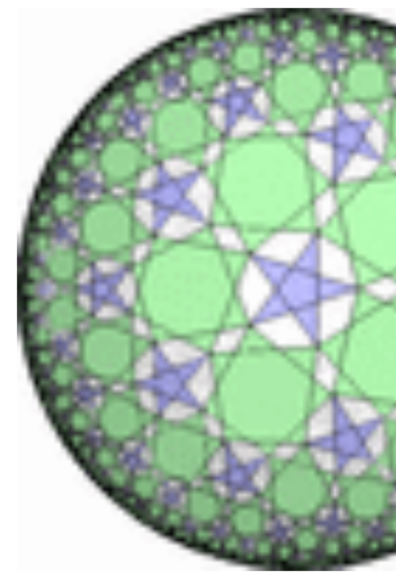


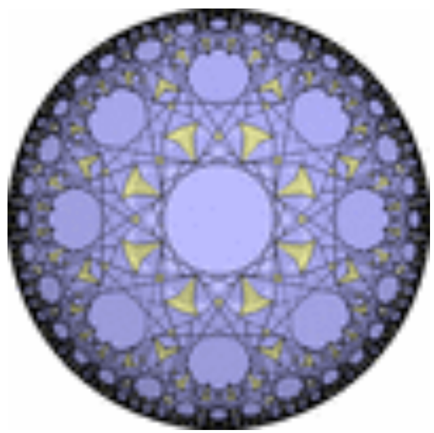


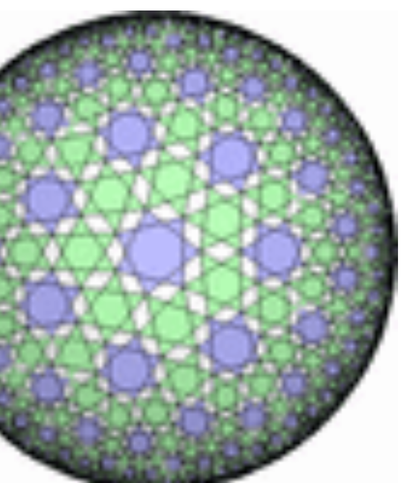










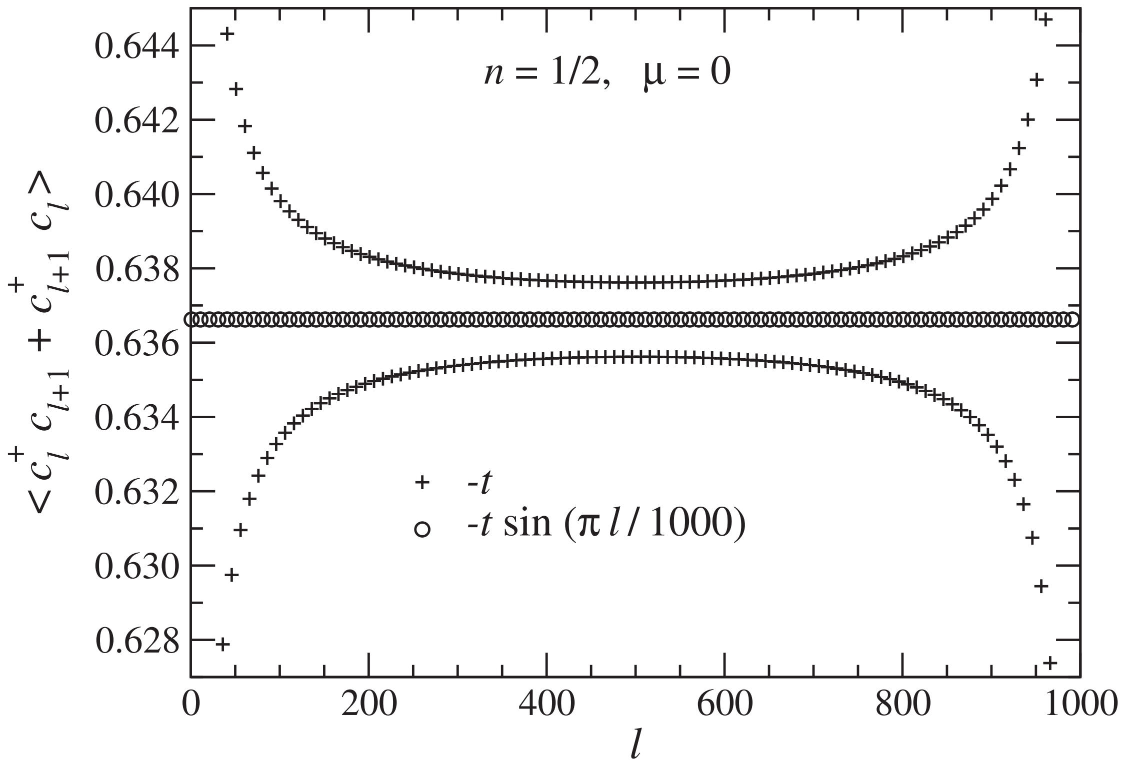


まあ、今日はこの辺で
失礼します。

A Generalization: Spherical Deformation

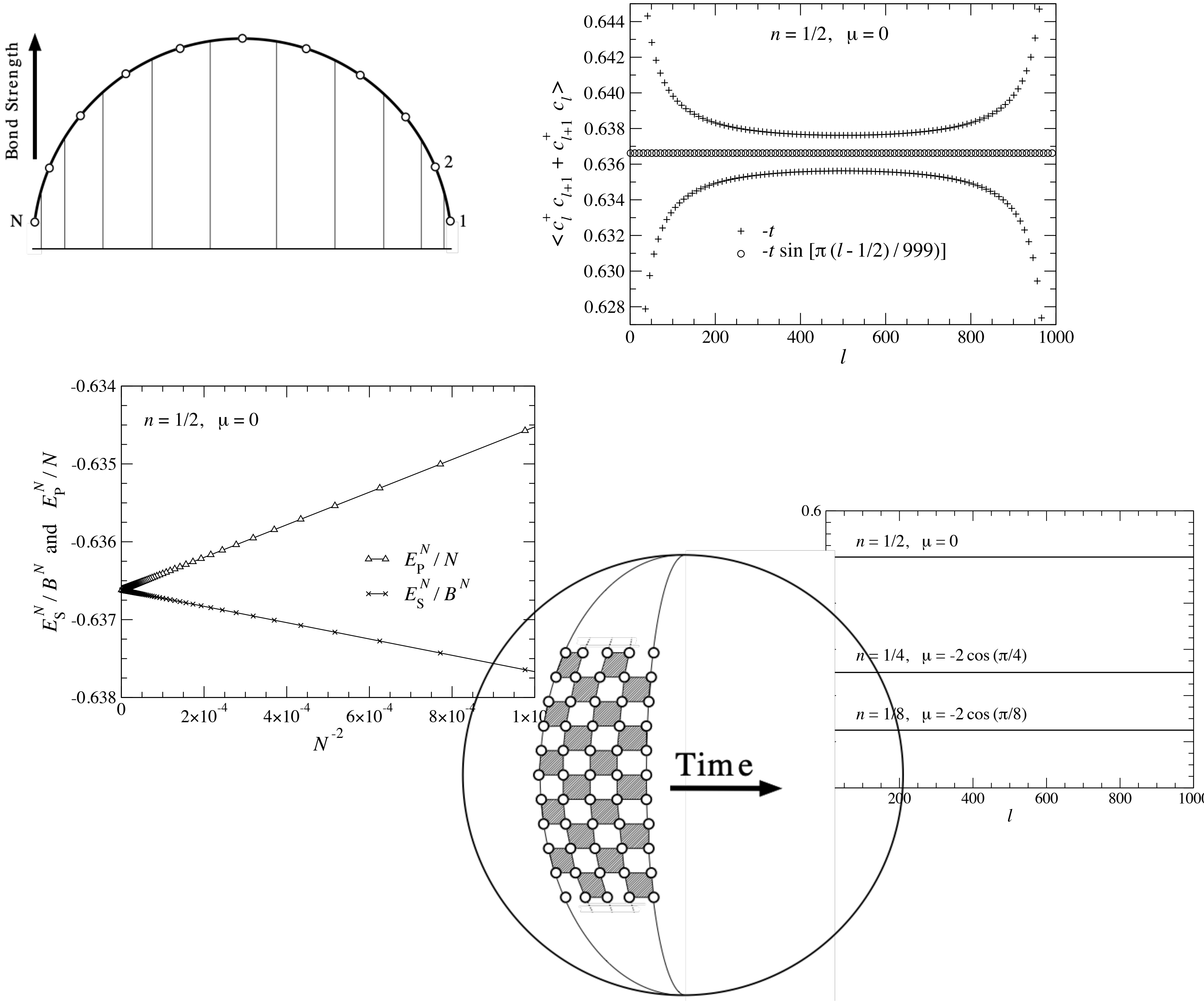
N-site tight binding Hamiltonian

$$\hat{H}_S = -t \;{}^{N-1}_{=1}\; \sin^2 \frac{\pi}{N} \;\; \hat{c}^{\dagger}\hat{c}_{+1} + \hat{c}^{\dagger}_{+1}\hat{c}$$



Boundary effect on the bond energy disappears completely!

A system under Open Boundary Condition gives data as efficient as those under Periodic Boundary Condition, under the spherical deformation.



Bond Strength
N
1
2
n = 1/2, μ = 0
< c_l^+ c_{l+1} + c_{l+1}^+ c_l >
-t
-t sin [π (l - 1/2) / 999)]
l
E_S^N / B^N and E_P^N / N
E_P^N / N
E_S^N / B^N
N^-2
n = 1/4, μ = -2 cos (π/4)
n = 1/8, μ = -2 cos (π/8)
Time

ディリクレ・ノイマン混合境界条件とhalf SSD

桂 法称 (東京大学 物理学専攻)

THE UNIVERSITY OF TOKYO 東京大学

Institute for Physics of Intelligence

Acknowledgment:
奥西巧一（新潟大）

Outline

Introduction

- What is SSD? What are special about SSD?
- Free-fermion models
- Results

Boundary conditions in lattice models

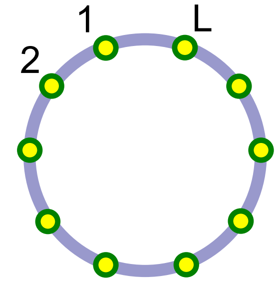
Half-SSD

Summary

What is SSD (sine-square deformation)?

■ Setup and definitions

Consider a lattice model on a chain of length *L*. (PBC imposed)



- Uniform Hamiltonian $\mathcal{H}_0 = \sum_{j=1}^{L} h_j + \sum_{j=1}^{L} h_{j,j+1}$
- Chiral Hamiltonian

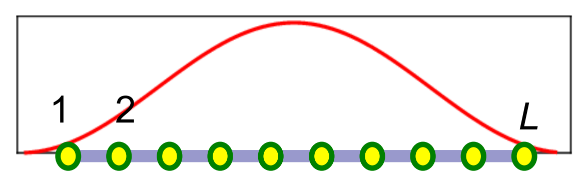
$$\mathcal{H}_\pm = \sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} h_j + \sum_{j=1}^{L} e^{\pm i\delta j} h_{j,j+1} \qquad \left(\delta = \frac{2\pi}{L}\right)$$

- SSD Hamiltonian

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

Gendiar *et al.*, *PTP* (2009-2010)
Hikihara, Nishino, *PRB* (2011)

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

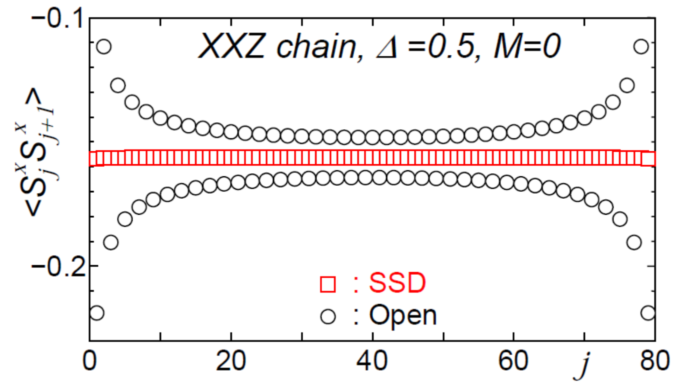


Ex.) Heisenberg chain

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\sum_{j=1}^{L}\left(1 - \frac{1}{2}e^{i\delta j} - \frac{1}{2}e^{-i\delta j}\right)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} = \sum_{j=1}^{L} \sin^2\left(\frac{\pi}{L} j\right)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

What are special about SSD?

- Suppression of boundary effects
  - Negligible Friedel oscillation
    Uniform g.s. correlations
  - Observed in ***1D critical systems***
    XXZ, Hubbard, Kondo-lattice, …
    Shibata, Hotta, *PRB* (2011)



Hikihara, Nishino, *PRB* **83**, 060414 (2011)

- Scaling of entanglement entropy

$$\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3} \ln\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + s_1$$

$$\mathcal{S}^{\mathrm{SSD}} \simeq \mathcal{S}^{\mathrm{PBC}}$$

- Wavefunction overlap

Overlap between the g.s. of systems with PBC and SSD is almost 1.

$$\langle \Psi_{\mathrm{SSD}} | \Psi_{\mathrm{PBC}} \rangle \simeq 1$$

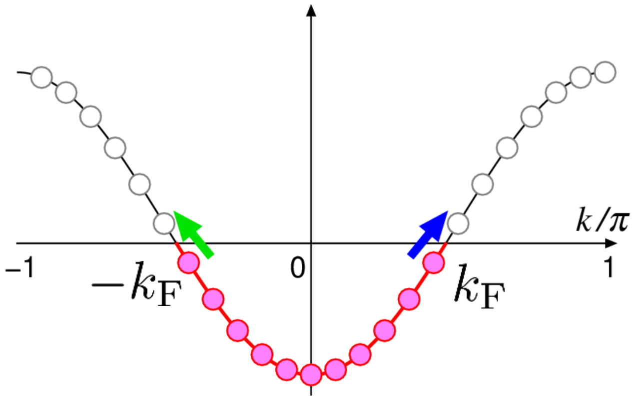
- Rigorous proof for ***free-fermion models*** (XY, quantum Ising, …)
- CFT interpretation: H.K., *JPA* **44**, 252001; **45**, 115003 (2011)

Free fermion chain with SSD (1)

- Uniform Hamiltonian

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} c_j^\dagger c_j$$

$c_j/c_j^\dagger$: annihilation/creation of fermion at *j*.



Fourier.tr.

$$\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k \qquad \epsilon(k) = -2t\cos k - \mu$$

G.S. of $\mathcal{H}_0$: Fermi sea ($\epsilon(k) < 0$ occupied) $\quad \mathcal{H}_0|\mathrm{FS}\rangle = E_g|\mathrm{FS}\rangle$

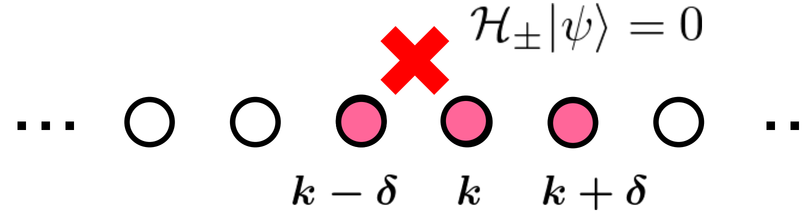
- Chiral Hamiltonian

$$\mathcal{H}_\pm = -t\sum_{j=1}^{L} e^{\pm i\delta j}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} c_j^\dagger c_j \quad \left(\delta = \frac{2\pi}{L}\right)$$

Momentum rep.

$$\mathcal{H}_\pm = e^{\mp i\delta/2}\sum_k \epsilon(k\mp\delta/2) c_k^\dagger c_{k\mp\delta}$$

$\mathcal{H}_\pm|\psi\rangle = 0$



If $\epsilon(k_\mathrm{F}+\delta/2) = \epsilon(-k_\mathrm{F}-\delta/2) = 0$, then $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0.$ ($\because$ $(c_k^\dagger)^2 = 0$)

Free fermion chain with SSD (2)

■ SSD Hamiltonian

$$\mathcal{H}_{\mathrm{SSD}} = -t \sum_{j=1}^{L-1} \sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu \sum_{j=1}^{L-1} \sin^2\left[\frac{\pi}{L}\left(j - \frac{1}{2}\right)\right] c_j^\dagger c_j$$

In terms of $\mathcal{H}_0$ & $\mathcal{H}_\pm$, $\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$

Fermi sea is annihilated by chiral Hamiltonians!

$$\mathcal{H}_\pm |\mathrm{FS}\rangle = 0$$

$$\mathcal{H}_{\mathrm{SSD}}|\mathrm{FS}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \cancel{\frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)}\right]|\mathrm{FS}\rangle = \frac{E_g}{2}|\mathrm{FS}\rangle$$

Fermi sea is an exact eigenstate of $\mathcal{H}_{\mathrm{SSD}}$!

■ Uniqueness of the ground state

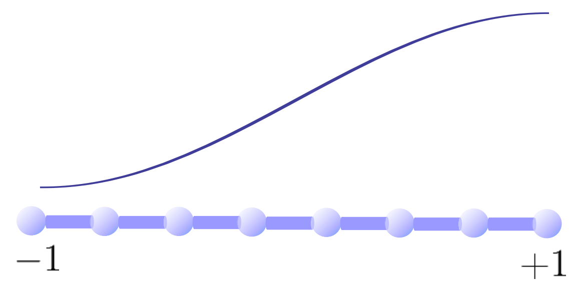
Fermi sea is ***the unique*** g.s. of $\mathcal{H}_{\mathrm{SSD}}$. $\mathcal{H}_0$ & $\mathcal{H}_{\mathrm{SSD}}$ share the same g.s.

Outline of proof)
Free-fermion chain → XY spin chain (via Jordan-Wigner)
Perron-Frobenius thm tells: (i) the ground state of $\mathcal{H}_{\mathrm{SSD}}$ is unique.
(ii) it has nonvanishing overlap with $|\mathrm{FS}\rangle$.

Results

- Any open/open correspondence?
- Yes, for free-fermion chain!
- But we need to add boundary potentials

$$H_0 = -\sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + \mathrm{h.c.}) - n_1 + n_L$$



$$H_{\mathrm{half}} = -\sum_{j=1}^{L-1} \sin^2\left(\frac{\pi j}{2L}\right)(c_j^\dagger c_{j+1} + \mathrm{h.c.}) + n_L$$

- H_0 and H_{half} share the same ground state

But why?

Outline

Introduction

Boundary conditions in lattice models

- Tight-binding model with boundary potential
- Analytically solvable cases, eigenfunctions

Half-SSD

Summary

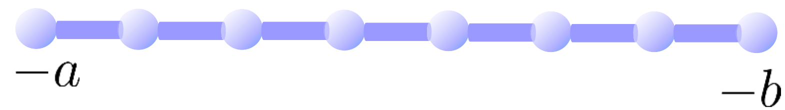
Tight-binding chain with boundary potential

■ Hamiltonian

$$H_0(a,b) = -\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1}) - a\, n_1 - b\, n_L$$

$c_j, c_j^\dagger$: spinless fermion ops.,

$n_j = c_j^\dagger c_j$: number op.



■ Tri-diagonal matrix

$$H_0(a,b) = -\boldsymbol{c}^\dagger\, \mathcal{T}(a,b)\, \boldsymbol{c}, \quad \boldsymbol{c}^\dagger = (c_1^\dagger, \ldots, c_L^\dagger)$$

- The \`hopping' matrix *T* determines the 1-particle spectrum

$$\mathcal{T}(a,b) = \begin{pmatrix} a & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & 1 & b \end{pmatrix}$$

- Eigenvalue problem

$$\mathcal{T}(a,b)\boldsymbol{v} = \lambda \boldsymbol{v}$$

Analytically tractable?

Yes, for special (*a*, *b*)

List of exact solutions

- *a* = *b* = 0: Fixed-Fixed BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{L+1}\right), \quad m = 1, 2, ..., L$$

Martin-Delgado, Sierra, *Phys. Lett. B* **364**, 41 (1995)

- *a* = *b* = 1: Free-Free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{L}\right), \quad m = 0, 1, ..., L-1$$

- *a* = 0, *b* = 1: Fixed-Free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{2L+1}\right), \quad m = 1, 3, 5, ..., 2L-1$$

- *a* = 1, *b* = -1: Free-Anti-free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{2L}\right), \quad m = 1, 3, 5, ..., 2L-1$$

- *a* = *q*, *b* = 1/*q*: Saleur (proceedings, 1989)

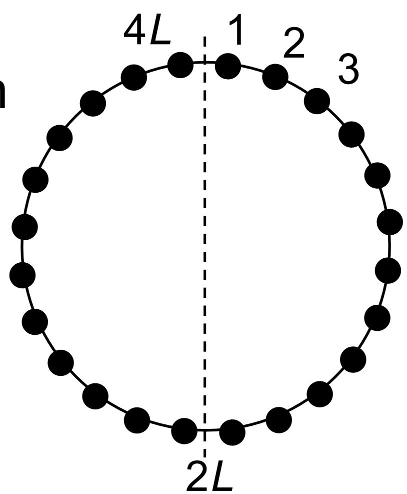
$$q + q^{-1}, \quad \lambda_m = 2\cos\left(\frac{m\pi}{L}\right) \quad m = 1, 2, ..., L-1$$

Appendix in HK, Schuricht, Takahashi, *PRB* **92**, 115137 (2015)

How to get eigenfunctions

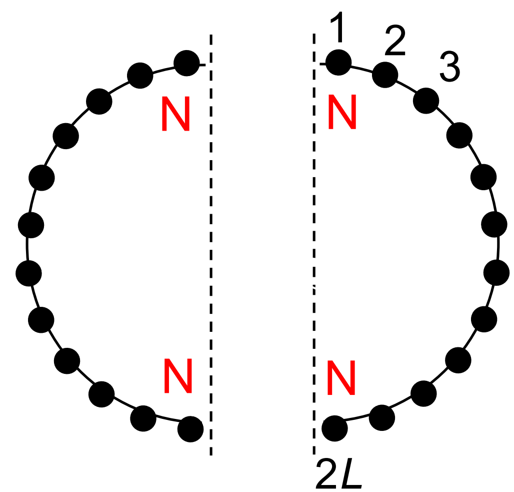
- Orbifolding?

Tight-binding on a periodic chain of length 4L



Neumann BC.:

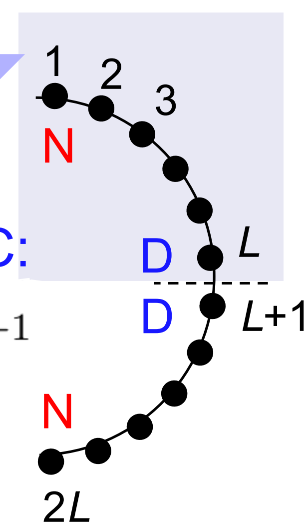
$$\psi_1 = \psi_{4L}, \quad \psi_{2L} = \psi_{2L+1}$$



One can get eigenfunctions of T(1,-1) from the plane wave solutions on a periodic ring!

Dirichlet BC:

$$\psi_L = -\psi_{L+1}$$



- Eigenfunctions

$$\psi_j^{(k)} = \sqrt{\frac{2}{L}} \cos\left[\frac{\pi(2j-1)(2k-1)}{4L}\right]$$

$$1 \leq j \leq L, \quad 1 \leq k \leq L$$

Outline

Introduction

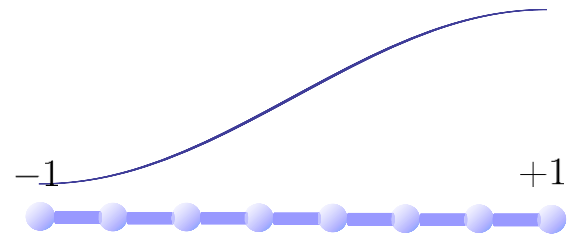
Boundary conditions in lattice models

Half-SSD

- What is half-SSD?
- Open/Half-SSD correspondence
- Self-duality, commuting property

Summary

What is half SSD?



■ Setup

- Open-chain Hamiltonian

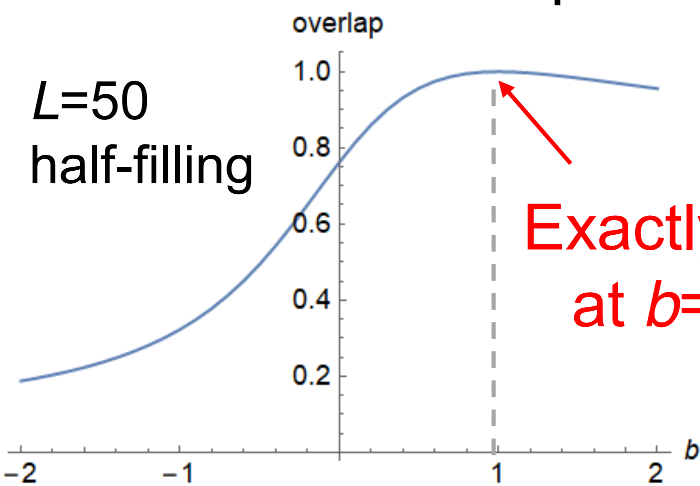
$$H_0(1,-1) = -\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + \text{h.c.}) - n_1 + n_L$$

- Half-SSD Hamiltonian with boundary potetial

$$H_{\text{half}}(b) = -\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi j}{2L}\right)(c_j^\dagger c_{j+1} + \text{h.c.}) + b\,n_L$$

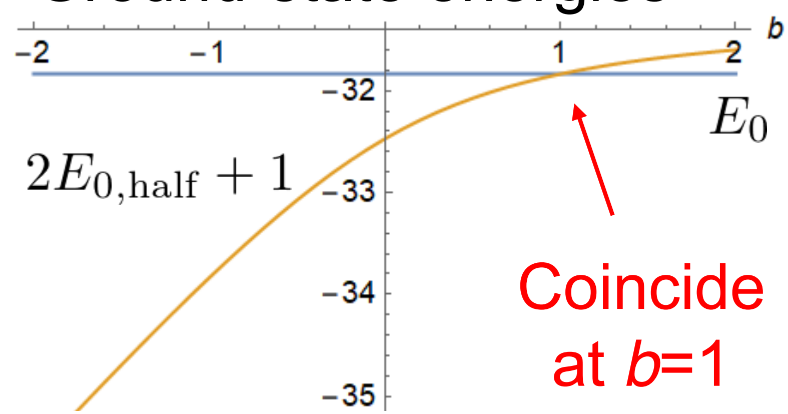
■ Ground-state overlap

L=50
half-filling



Exactly 1
at *b*=1

■ Ground-state energies



Coincide
at *b*=1

Open/half-SSD correspondence!

What happens at *b*=1?

- $H_{\rm half}$ in the H_0-diagonal basis (L: even)

$$H_{\rm half}(b) = \boldsymbol{c}^\dagger \mathcal{H}_{\rm half}(b)\, \boldsymbol{c},$$

$$\mathsf{H}_{k,l} = \langle \psi^{(k)}, \mathcal{H}_{\rm half}(1)\psi^{(l)} \rangle$$

Eigenfunctions of $T(1,-1)$

$$\psi_j^{(k)} = \sqrt{\frac{2}{L}} \cos\left[\frac{\pi(2j-1)(2k-1)}{4L}\right]$$

- Diagonal matrix elements

$$\mathsf{H}_{k,k} = -\cos\left[\frac{\pi(2k-1)}{2L}\right] + \frac{1}{2}(\delta_{k,1} + \delta_{k,L}), \quad (k = 1, \ldots, L)$$

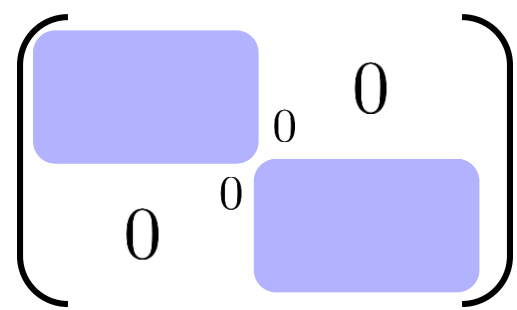
- Off-diagonal elements

$$\mathsf{H}_{k,k+1} = \mathsf{H}_{k+1,k} = \frac{1}{2}\cos\left(\frac{\pi k}{L}\right), \quad (k = 1, \ldots, L-1)$$

The other elements are all zero.

Clearly

$$\mathsf{H}_{L/2,L/2+1} = \mathsf{H}_{L/2+1,L/2} = 0.$$



Fermi sea of H_0 is the g.s. of $H_{\rm half}$!

Self-dual Hamiltonian

■ Interpolation between H_{half} and H_0

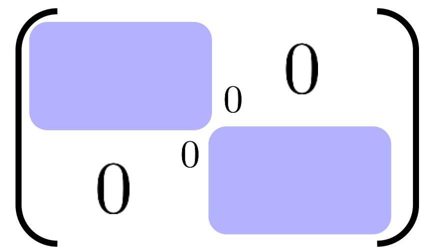
$$H_{\text{half}}(1) = \frac{1}{2}H_0(1,-1) + \frac{1}{2}\left[\underline{\sum_{j=1}^{L-1}\cos\left(\frac{\pi j}{L}\right)(c_j^\dagger c_{j+1} + \text{h.c.}) + n_1 + n_L}\right]$$

H_1

- 1-parameter Hamiltonian

$$H(\alpha) = \frac{1}{2}(H_0(1,-1) + \alpha H_1)$$

N.B.: $H(0) = \frac{1}{2}H_0(1,-1),\ H(1) = H_{\text{half}}(1)$

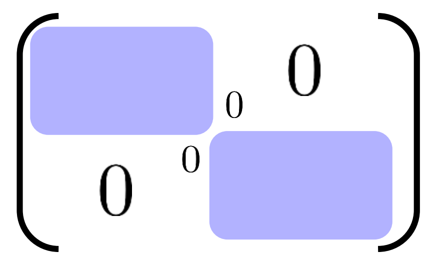
$\text{H}_1 =$ 

■ Self-duality of H_1

- Eigen-operators of H_0 $\quad \tilde{c}_k = \sum_{j=1}^{L}\psi_j^{(k)}c_j,\quad \tilde{c}_k^\dagger = \sum_{j=1}^{L}\psi_j^{(k)}c_j^\dagger$

- Eigen-space Hamiltonian (H_1)

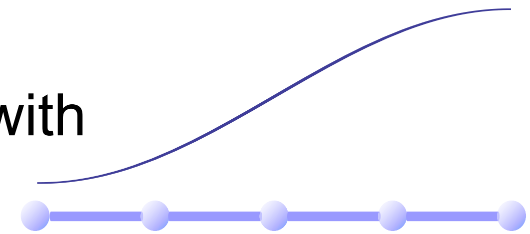
$$H_1 = \sum_{j=1}^{L-1}\cos\left(\frac{\pi j}{L}\right)(\tilde{c}_j^\dagger \tilde{c}_{j+1} + \text{h.c.}) + \tilde{c}_1^\dagger \tilde{c}_1 + \tilde{c}_L^\dagger \tilde{c}_L$$



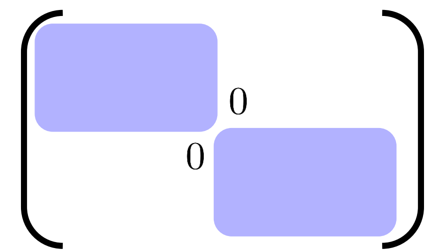
Takes the same form as H_1 in real-space!

Summary

- Studied tight-binding chain with half-SSD



- The models shares the same ground state with the tight-binding chain with special b.c.
- The b.c. = mixed Dirichlet-Neumann b.c.
- Decoupling structure in the `eigen-space'



Future directions

- Extension to finite chemical potential
- Field theory: bosonization, CFT, …
- Algebra? Anything to do with modular *S*-matrix? $\psi_j^{(k)} = S_j^k \ (?)$
- Extension to other boundary conditions?
  Other mixed b.c. What about Robi b.c.? $(\alpha\psi + \beta\psi')\big|_{\partial\Omega} = 0$
- What are they good for?

タキオン真空とサイン二乗変形

高橋　智彦

奈良女子大学 素粒子論研究室

2019 年 7 月 11 日
(理化学研究所, 和光)

PTEP, 2018, 12, 123B04 [arXiv:1809.01885 [hep-th]]
with Isao Kishimoto, Tomomi Kitade

はじめに

"Dipolar quantization and the infinite circumference limit of two-dimensional conformal field theories",
Nobuyuki Ishibashi, Tsukasa Tada
Int. J. Mod. Phys. A 31, 1650170 (2016) [arXiv:1602.01190v1]

The present formulation was also partially guided by previous approaches in the study of string field theory (SFT) [37, 38]*. It would be interesting if one can find more direct connections between the present result and SFT treatment, especially in the context of understanding the transition between open and closed strings* [39]*.*

[37] M. Kiermaier, A. Sen and B. Zwiebach, JHEP0803, 050 (2008) [arXiv:0712.0627 [hep-th]].
[38] T. Takahashi and S. Zeze, Prog. Theor. Phys.110, 159 (2003) [hep-th/0304261].
[39] T. Takahashi, Prog. Theor. Phys. Suppl.188, 163 (2011).

理研研究会「サイン 2 乗変形（SSD）とその周辺」
日時: 2017 年 6 月 30 日 10:30 - 17:20
会場: セミナー室 (160 号室)

印象に残ったこと

SSD mechanism

· $\left\langle v_{\mathrm{SSD}} \,\middle|\, v_{\mathrm{PBC}} \right\rangle = 1$

⇒ 帰ってから Mathematica で確かめた! (驚きました)

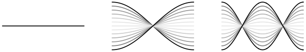
· $\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-), \qquad \mathcal{H}_\pm |0\rangle = 0$

· $H = \frac{1}{2}L_0 - \frac{1}{4}(L_1 + L_{-1}), \qquad L_0 |0\rangle = L_{\pm 1} |0\rangle = 0$

⇒ 異なった言い方ができるはず?

ボゾン型開弦のタキオン

開弦の固有状態



	状態	$(\text{質量})^2$	スピン	成分数
タキオン	$\|p\rangle$	$-\dfrac{1}{\alpha'}$	0	1
ベクトル粒子	$\alpha^i_{-1}\|p\rangle$	0	1	$D-2=24$
テンソル粒子	$\alpha^i_{-1}\alpha^j_{-1}\|p\rangle$ $(p^i\alpha^j_{-2}-p^j\alpha^i_{-1})\|p\rangle$	$+\dfrac{1}{\alpha'}$	2	$\frac{D(D-2)}{2}=276$

(臨界次元: $D=26$)

$$p^{j_1}\cdots p^{j_M}\alpha^{i_1}_{-n_1}\cdots\alpha^{i_N}_{-n_N}|p\rangle\,,\qquad M^2=\Big(\sum_{k=1}^{N}n_k-1\Big)/\alpha'$$

$$\left[\,\alpha^i_n,\,\alpha^j_m\,\right]=n\delta_{n+m,0}\delta^{i,j}$$

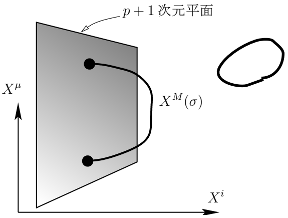
D ブレーン

Polchinski '95

D ブレーンとは、開弦の端点がくっつく $p+1$ 次元の超曲面

($p+1 \leq 26$)

- $X^\mu(\sigma)$ $(\mu = 0, \cdots, p)$: ノイマン境界条件
  $X^i(\sigma)$ $(i = p+1, \cdots, 25)$: ディリクレ境界条件
- D ブレーン自身が力学的な対象
- 開弦は D ブレーンのゆらぎを表す
- タキオンの存在はボゾン型 D ブレーンの不安定性を示す
- バルク時空には閉弦が存在する



弦の場の理論 (String Field Theory)

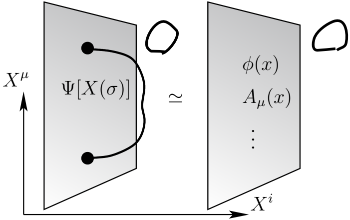
場 $\phi(x)$ を拡張した弦の場 $\Psi[X(\sigma)]$ を力学変数とする理論

Witten ’86

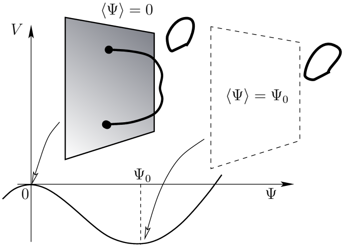
$$S[\Psi] = \frac{1}{g^2}\int\left(\frac{1}{2}\Psi * Q_B\Psi + \frac{1}{3}\Psi*\Psi*\Psi\right)$$

$$= \frac{1}{g^2}\int d^{p+1}x\left\{-\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2\alpha'}\phi^2 - \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{3}\kappa^3\left(e^{\log\kappa\,\alpha'\partial^2}\phi\right)^3 + \cdots\right\}$$

$$(\kappa = 3\sqrt{3}/4)$$



タキオン真空



Sen '99, Sen-Zwieback '00, $\cdots$

T.T-Tanimoto '02, Kishimoto-T.T '02, $\cdots$

Schnabl '05, Erler-Scnabl '09, T.T-Zeze '03, $\cdots$

Ishibashi '14, Kishimoto-Masuda-T.T '14, $\cdots$

- 弦の場の理論には安定なタキオン真空が存在
- タキオン真空では、D ブレーンが消滅
  (Ψ_0 に対するエネルギーが、D ブレーンのエネルギーを相殺)
- タキオン真空では、開弦の場のゆらぎはゲージ自由度となる
  (D ブレーンのゆらぎの自由度が消えている)

タキオン真空上での" 開弦" のハミルトニアン

Takahashi ('03), Takahashi-Zeze ('03)

$$
\begin{aligned}
L' &= \frac{1}{2}L'_0 - \frac{1}{4}(L'_2 + L'_{-2}) + \frac{3}{2} \\
&= \oint \frac{dz}{2\pi i}\frac{-1}{4z}(z^2-1)^2\, T'(z) + \cdots \\
&= \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} \sin^2\sigma\, T'(\sigma) + \cdots
\end{aligned}
$$

タキオン真空上の開弦のハミルトニアンにサイン二乗変形が表れている

$\Leftarrow$　石橋-多田の指摘!

SSD

開いた一次元系をサイン二乗変形した系の基底状態は、周期的境界条件を課した系の基底状態と一致している。

SFT

タキオン真空では、D ブレーンが消滅しているので開弦のタキオン状態は基底状態ではなくなり、バルクに閉弦が存在するので閉弦の基底状態がこの系の基底状態になるはずだ!

予 想

サイン二乗変形 $\simeq$ タキオン真空

1 次元離散フェルミオン系

"Field Theories of Condensed Matter Physics", E. Fradkin

1 次元フェルミオン系

$$\hat{H}_{\mathrm{O}} \quad = \quad -t\sum_{l=0}^{N}\left(\hat{c}_l^{\dagger}\hat{c}_{l+1}+\hat{c}_{l+1}^{\dagger}\hat{c}_l\right).$$

$\hat{c}_l$ を

$$\hat{c}_l \equiv e^{i\frac{\pi}{2}l}a_l \tag{3.1}$$

と変換すると、ハミルトニアンは

$$\hat{H}_{\mathrm{O}} = -it\sum_{l=1}^{N}a_l^{\dagger}(a_{l+1}-a_{l-1}), \qquad (a_0 = a_{N+1} = 0)$$

$$\{a_l,\, a_{l'}^{\dagger}\} = \delta_{l,l'}, \quad \{a_l,\, a_{l'}\} = \{a_l^{\dagger},\, a_{l'}^{\dagger}\} = 0$$

さらに、

$$\varphi_l^1 \equiv \frac{1}{\sqrt{2}}(a_l + a_l^\dagger), \quad \varphi_l^2 \equiv \frac{1}{\sqrt{2}i}(a_l - a_l^\dagger),$$

として、エルミート演算子 $\varphi_l^\mu \quad (\mu = 1, 2)$ を導入すると、

$$\hat{H}_{\mathrm{O}} = -it \sum_{l=0}^{N} \left(\varphi_l^1 \varphi_{l+1}^1 + \varphi_l^2 \varphi_{l+1}^2 \right) \quad (\varphi_0^\mu = \varphi_{N+1}^\mu = 0)$$

$$\{\varphi_l^\mu, \varphi_{l'}^\nu\} = \delta_{l,l'} \delta^{\mu\nu}$$

のように、独立な演算子 φ_l^1 と φ_l^2 に対して対称な形に書きなおすことができる。

この系の連続極限について考えていこう。一つの自由度を抜き出し、それを φ_l として考える。
まず、その 1 自由度に対するハミルトニアンを、偶数サイトの演算子と奇数サイトの演算子に分けて

$$\hat{H}_{\mathrm{O}} = -\frac{it}{2} \sum_{k=1}^{\left[\frac{N+1}{2}\right]} \left\{ \varphi_{2k-1}(\varphi_{2k} - \varphi_{2k-2}) + \varphi_{2k}(\varphi_{2k+1} - \varphi_{2k-1}) \right\}$$

と書く。ここで、$\xi_k = \varphi_{2k-1}, \quad \eta_k = \varphi_{2k}$ という演算子を導入すると、反交換関係は

$$\{\xi_k, \xi_{k'}\} = \delta_{k,k'}, \quad \{\eta_k, \eta_{k'}\} = \delta_{k,k'}, \quad \{\xi_k, \eta_{k'}\} = 0$$

であり、ハミルトニアンは

$$\hat{H}_{\mathrm{O}} = -\frac{it}{2} \sum_{k=1}^{\left[\frac{N+1}{2}\right]} \left\{ \xi_k(\eta_k - \eta_{k-1}) + \eta_k(\xi_{k+1} - \xi_k) \right\}$$

となる。

連続極限をとると、$N \to \infty$ の極限をとると、連続極限をとったハミルトニアンが

$$\hat{H}_{\mathrm{O}} = -\frac{i}{2\pi}\int_0^\pi \left(\xi\eta' + \eta\xi'\right) d\sigma$$

となり、反交換関係が

$$\begin{aligned} &\{\xi(\sigma), \xi(\sigma')\} = \pi\delta(\sigma - \sigma'), \quad \{\eta(\sigma), \eta(\sigma')\} = \pi\delta(\sigma - \sigma'), \\ &\{\xi(\sigma), \eta(\sigma')\} = 0, \end{aligned}$$

となる。
境界条件は、N を偶数に保ちながら極限をとる場合と、奇数に保ちながら極限をとる場合とで異なってくる。
N が偶数の場合、格子上の変数の境界条件は $\varphi_0 = \varphi_{N+1} = 0$ であるが、$N+1$ が奇数となるため、連続変数の境界条件が

$$\eta(0) = 0, \qquad \xi(\pi) = 0$$

となる。同様に考えて、N が奇数の場合の境界条件は

$$\eta(0) = 0, \qquad \eta(\pi) = 0$$

となる。

弦理論との対応をより明確に見るために、

$$\psi_\pm \equiv \frac{1}{\sqrt{2}}(\xi \pm \eta)$$

と定義すると、ハミルトニアンが

$$\hat{H}_{\mathrm{O}} = \frac{i}{2\pi}\int_0^\pi \left(\psi_+\partial_\sigma\psi_+ - \psi_-\partial_\sigma\psi_-\right) d\sigma$$

となり、反交換関係が

$$\begin{aligned}&\{\psi_+(\sigma), \psi_+(\sigma')\} = \pi\delta(\sigma-\sigma'), \quad \{\psi_-(\sigma), \psi_-(\sigma')\} = \pi\delta(\sigma-\sigma'),\\ &\{\psi_+(\sigma), \psi_-(\sigma')\} = 0,\end{aligned}$$

となる。
これは、２次元マヨラナフェルミオン系に対するハミルトニアンであり、開いた超弦理論のフェルミオン部分に対応していることがわかる。
境界条件は、N が偶数の場合、

$$\psi_+(0) = \psi_-(0), \quad \psi_+(\pi) = -\psi_-(\pi)$$

となり、Neveu-Schwarz 型の境界条件である。N が奇数の場合は、

$$\psi_+(0) = \psi_-(0), \quad \psi_+(\pi) = \psi_-(\pi)$$

となり、Ramond 型の境界条件であることがわかる。

サイン二乗変形した系の連続極限

$$\hat{H}_{\mathrm{O}} = \frac{i}{2\pi}\int_0^\pi \sin^2\sigma\ (\psi_+\partial_\sigma\psi_+ - \psi_-\partial_\sigma\psi_-)\,d\sigma$$

N が偶数の場合、Neveu-Schwarz 型の境界条件

$$\psi_+(0) = \psi_-(0), \quad \psi_+(\pi) = -\psi_-(\pi)$$

N が奇数の場合、Ramond 型の境界条件

$$\psi_+(0) = \psi_-(0), \quad \psi_+(\pi) = \psi_-(\pi)$$

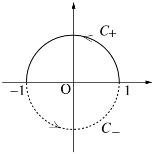
境界条件があるために、$\psi_+(\sigma)$ と $\psi_-(\sigma)$ が独立な演算子とならない！
$\Rightarrow$ 開弦の特徴！

なぜ、周期的境界条件をもつ系（閉弦系）を記述するのか?

Open string Hamiltonian

The Hamiltonian of an open string is given by

$$H_{\rm O}=\int_{C_+}\frac{dz}{2\pi i}\,zT(z)+\int_{C_-}\frac{dz}{2\pi i}\,zT(z),$$



$T(z)$: the energy-momentum tensor.

Each term corresponds to Hamiltonians of left and right moving modes, respectively, but they do not commute with each other due to open boundary conditions on $T(z)$.

The Hamiltonian is given by the zeroth component of the Virasoro operators: L_0. So, we do not encounter antiholomorphic Virasoro operators in the open string system.

Sine-squre-like deformation

Here, we consider the deformed Hamiltonian:

$$H_g = H_g^+ + H_g^-, \quad H_g^\pm = \int_{C_\pm} \frac{dz}{2\pi i}\, g(z) T(z),$$

where $g(z)$ is a holomorphic function satisfying $g(\pm 1) = \partial g(\pm 1) = 0$.

H_g^+ and H_g^- are left and right moving modes of H_g.

The simplest example of $g(z)$ is given by

$$g(z) = -\frac{1}{4z}(z^2-1)^2.$$

If we change the variable as $z = \exp(i\theta)$, the weighting function in H_g is changed to $z^{-1}g(z) = \sin^2\theta$. Hence, the deformed Hamiltonian provides a sort of generalization of the SSD Hamiltonian. In this sense, we call it the sine-square-like deformation, or SSLD for short.

$T(z)$ is expanded by holomorphic Virasoro operators only:

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}.$$

By using this expansion form and the Virasoro algebra, we can obtain a commutation relation of $T(z)$:

$$[T(z), T(z')] = -(\, T(z) + T(z')\,)\, \partial\delta(z, z') - \frac{c}{12}\partial^3\delta(z, z'),$$

where c is the central charge of $T(z)$.

By this equation, we can calculate the commutation relation between H_g^+ and H_g^-.

The important point is that surface terms appear in the calculation as a result of derivatives of the delta function and these terms include a singular factor $\delta(\pm 1, \pm 1)$.

However, the singular surface terms turn out to vanish due to the factors $g(\pm 1)$ and $\partial g(\pm 1)$, which are set to zero in the definition of H_g.

As a result, we find

$$\left[\, H_g^+,\, H_g^- \,\right] = 0$$

and then the deformed system is decomposed into the left and right moving parts as in periodic systems.

Accordingly, it is concluded that the deformed system described by H_g corresponds not to an open string system, but to a closed string system, although the Hamiltonian is constructed by a single holomorphic energy-momentum tensor.

It should be noted that **the zeros of $g(z)$ and $\partial g(z)$ at open string boundaries cause the decoupling of the left and right moving sectors!**

Now, we will illustrate equal-time contours generated by the Hamiltonian for the simplest function

$$g(z) = -\frac{1}{4z}(z^2-1)^2.$$

with a focus on emergence of left and right moving sectors.

According to Ishibashi-Tada, we introduce the parameters, t and s, into the worldsheet generated by H_g:

$$t + is = \int^z \frac{dz}{g(z)} = \frac{2}{z^2-1},$$

where t denotes time and s parameterizes a string at a certain time.

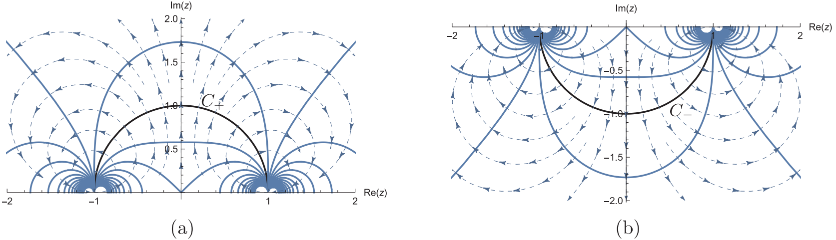
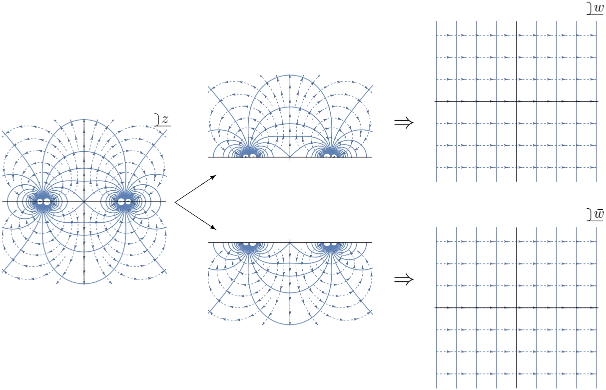


Figure: Equal-time contours on the z plane (solid lines). Dashed lines with arrows denote evolution of time t.

These contours have a remarkable feature that the string boundaries are fixed at $z = \pm 1$ during propagation of the string. One complex number $t + is$ corresponds to two points in the z plane.

Accordingly, we introduce a complex coordinate $w = t + is$ for the upper half z plane and $\bar{w} = t + is$ for the lower half plane.

By this mapping, the upper half plane corresponds to the whole w plane, and the lower half plane to the other $\bar{w}$ plane:



Hence, the equal-time contours by H_g lead us to the worldsheet which consists of two complex planes.

The two planes, w and $\bar{w}$, corresponding to the upper and lower half z planes are generated by the left and right moving Hamiltonian, H_g^+ and H_g^-, respectively.

Therefore, they can be regarded as holomorphic and antiholomorphic worldsheets of a closed string.

Now that we have obtained two decoupled Hamiltonians for the left and right moving sectors, we can construct two independent Virasoro operators according to Ishibashi-Tada:

$$\mathcal{L}_\kappa = \int_{C_+^t} \frac{dz}{2\pi i} g(z) f_\kappa(z) T(z), \qquad \tilde{\mathcal{L}}_\kappa = \int_{C_-^t} \frac{dz}{2\pi i} g(z) f_\kappa(z) T(z),$$

where $g(z)$ is the same function as that in the Hamiltonian H_g.
$f_\kappa(z)$ is defined by the differential equation

$$g(z) \frac{\partial}{\partial z} f_\kappa(z) = \kappa f_\kappa(z).$$

For a constant time t, C_+^t and C_-^t denote integral contours along the equal-time line on the upper and lower half z plane, respectively.

We should note again that $T(z)$ **including in $\mathcal{L}_\kappa$ and $\tilde{\mathcal{L}}_\kappa$ is the same energy-momentum tensor of the open string system.**

$\mathcal{L}_0$ and $\tilde{\mathcal{L}}_0$ provide the left and right moving parts of the Hamiltonian, that is, $\mathcal{L}_0 = H_g^+$ and $\tilde{\mathcal{L}}_0 = H_g^-$.

$\mathcal{L}_\kappa$ satisfies continuous Virasoro algebra:

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} \\ + \frac{c}{12}\int_{C_+^t} \frac{dz}{2\pi i}\left\{(\kappa - \kappa')\left(\frac{\partial^2 g}{\partial z^2} - \frac{1}{2g}\left(\frac{\partial g}{\partial z}\right)^2\right) + \frac{\kappa^3 - {\kappa'}^3}{2g}\right\} f_{\kappa+\kappa'}(z).$$

Ishibashi-Tada '16

The right moving sector of the Virasoro operator $\tilde{\mathcal{L}}_\kappa$ can be also defined by integration along the integration path on the lower half plane. Similarly, $\tilde{\mathcal{L}}_\kappa$ satisfies the continuous Virasoro algebra.

Moreover, since C_+^t and C_-^t have no intersections, $\mathcal{L}_\kappa$ and $\tilde{\mathcal{L}}_\kappa$ commute with each other:

$$[\mathcal{L}_\kappa,\, \tilde{\mathcal{L}}_{\kappa'}] = 0.$$

Thus, we have found the **two independent Virasoro algebras in a deformed open string system, which can be regarded as the Virasoro algebras for closed strings, that is, the holomorphic and antiholomorphic parts.**

Kishimoto, Kitade and T.T ('18)

Energy-momentum tensor

We define an operator at the tachyon vacuum:

$$\begin{aligned}\mathcal{T}(z) &\equiv e^{-h(z)}\{Q',\, b(z)\} \\ &= T(z) + \partial h(z)\, j_{\rm gh}(z) - (\partial h(z))^2 + \frac{3}{2} e^{-h(z)} \partial^2 e^{h(z)}.\end{aligned}$$

We find that $\mathcal{T}(z)$ satisfies the same OPE as $T(z)$ with zero central charge:

$$\mathcal{T}(y)\mathcal{T}(z) \sim \frac{2}{(y-z)^2}\mathcal{T}(z) + \frac{1}{y-z}\partial\mathcal{T}(z).$$

Here, it should be noted that $\mathcal{T}(z)$ includes not only operators but also a function in its form.

Since $h(z)$ is related to a coordinate frame of worldsheets, $\mathcal{T}(z)$ has an explicit dependence on the frame.

Virasoro algebra

By using $\mathcal{T}(z)$, we can define the continuous Virasoro operator at the tachyon vacuum:

$$\mathcal{L}_\kappa \equiv \int_{C_+} \frac{dz}{2\pi i} g(z) f_\kappa(z) \mathcal{T}(z), \qquad \tilde{\mathcal{L}}_\kappa \equiv \int_{C_-} \frac{dz}{2\pi i} g(z) f_\kappa(z) \mathcal{T}(z),$$

where the weighting function is related to $h(z)$ as $g(z) = z e^{h(z)}$.

Since $e^{h(z)}$ has second order zeros at $z = \pm 1$, $g(z)$ also has second order zeros at $z = \pm 1$.

These operators satisfy the holomorphic and antiholomorophic continuous Virasoro algebra for $c = 0$. ($\mathcal{L}_0 = H_+$ and $\tilde{\mathcal{L}}_0 = H_-$.)

By definition of $\mathcal{T}(z)$, these operators commute with $Q'_\pm$:

$$[Q'_\pm, \mathcal{L}_\kappa] = [Q'_\pm, \tilde{\mathcal{L}}_\kappa] = 0.$$

Thus, we have found the continuous Virasoro algebra at the tachyon vacuum.

まとめ

開弦系のサイン二乗変形が閉じた系を表すのは、ハミルトニアンのゼロ点のために、ハミルトニアンの右向き部分とハミルトニアンの左向き部分が可換になるためである

ハミルトニアンの左右分離の結果、左右独立の連続ビラソロ代数が現れる

開弦の場の理論におけるタキオン真空上では、開弦のハミルトニアンにサイン二乗変形が現れる。

その結果、タキオン真空上での開弦の場の理論に、閉弦理論がもつ対称性を見出すことができる。

対称性の役割の大きさを考えれば、開弦の場の理論によって、閉弦の力学を解析できる可能性がある！

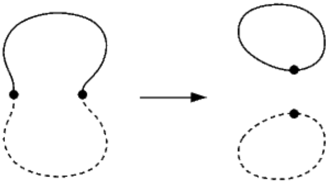


Figure: String pictures before and after SSLD. The solid and dashed lines correspond to holomorphic and antiholomorphic parts of a string. As a result of SSLD, open string boundaries (black dots) become joined and an open string divides to holomorphic and antiholomorphic strings.

Holographic duals of inhomogenous systems; Rainbow chain and SSD

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Aike Liu (UIUC → Caltech),
Masahiro Nozaki (U Chicago → RIKEN/Berkeley)

University of Chicago

July 10, 2019

Introduction

- Inhomogeneous quantum many-body systems (on a lattice):

$$H = \sum_i h_{i,i+1} \implies H = \sum_i f(x_i, x_{i+1}) h_{i,i+1}$$

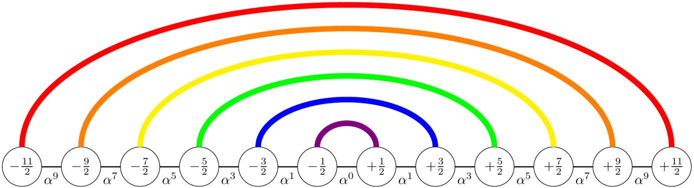
- Entanglement (Rindler) Hamiltonian: $f(x) = \frac{R^2 - x^2}{2R}$
- Sine-square deformation (SSD): $f(x) = \cos\frac{2\pi x}{L} + 1 = \sin^2\frac{\pi(x-L/2)}{L}$ [Gendiar-Krcmar-Nishino (08), Hikihara-Nishino (11), ...]
- Möbius evolution: $f(x) = \cos\frac{2\pi x}{L} + \sqrt{1 - const./L^2}$ [Ishibashi-Tada (15-16); Okunishi (16); Wen-SR-Ludwig (16)]
- Rainbow chain: $f(x) = e^{-h|x|}$

Rainbow chain

- [Vitagliano-Riera-Latorre (10), Ramirez-Rodriguez-Laguna-Sierra (14)]

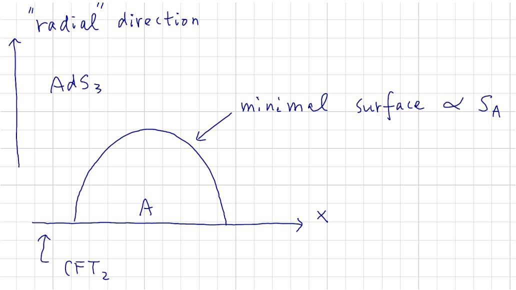
$$H = -c^\dagger_{1/2} c_{1/2} - \sum_{i=1/2}^{L-3/2} e^{-hi} \left[c^\dagger_i c_{i+1} + c^\dagger_{-i} c_{-i-1} \right]$$

- Concentric singlet formation
- Volume law entanglement for the half-chain partition: $S_A \sim L$.
- In the continuum, CFT on AdS_2.
  [Rodriguez-Laguna-Dubail-Ramirez-Calabrese-Sierra (16)]



- In this talk, I will develop holographic descriptions
- Of particular interest: the scaling of the entanglement entropy (at zero and finite T).

AdS/CFT, AdS_3/CFT_2 in particular



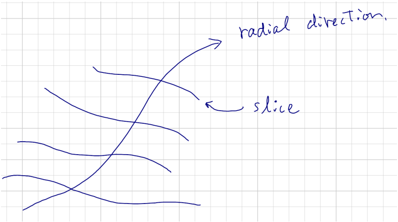
- Gravity in bulk AdS $\Leftrightarrow$ CFT on ∂AdS
- "Radius" R of AdS $\Leftrightarrow$ central charge c: $c = 3R/(2G_N)$
- BTZ black hole $\Leftrightarrow$ finite T

AdS/CFT, AdS_3/CFT_2 in particular

- *Kinematical*: Any stuff determined solely by conformal symmetry in CFT should have geometric descriptions in AdS. E.g., entanglement entropy for a single interval
- *Dynamical*: Einstein gravity in AdS realizes large c CFT. E.g., operator content, mutual information.

Different time-evolution $\leftrightarrow$ Different foliations

- Key concept and strategy: foliations (slicing)
- (Boundary metric $\rightarrow$ Einstein equation $\rightarrow$ Bulk geometry)
- Why different coord systems lead to different physics? (Diffeo invariance?) $\Leftarrow$ CFT is defined on a cutoff surface



Example: Entanglement (Rindler) Hamiltonian

- Rindler coord. $(u > 0, -\infty < t' < \infty)$:

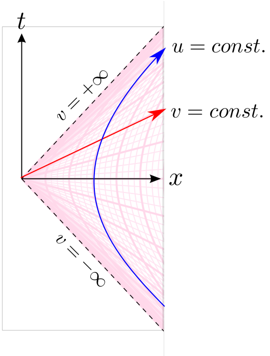
$$t = u \sinh(ht'), \quad x = u \cosh(ht')$$

- The half of the space(time) is inaccessible ("traced out"); the state is mixed at finite Unruh temperature $T = h/(2\pi)$.

- Metric:

$$ds^2_{Rindler} = -u^2 dt'^2 + du^2 = e^{2hx'}(-dt'^2 + dx'^2)$$

(Tortoise coordinate by $u =: h^{-1} e^{hx'}$.)



[Figures: Wikipedia]

Example: Entanglement (Rindler) Hamiltonian

- The Rindler Hamiltonian,

$$H_{Rindler} = \int_0^\infty du\, u\, \mathcal{H}(u).$$

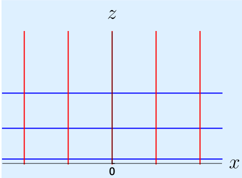
- Bulk metric:

$$\begin{aligned} ds^2_{AdS_3} &= R^2 \frac{dz^2 + dx^2 - dt^2}{z^2} \\ &= \frac{R^2}{z^2} \left[(-u^2 dt'^2 + du^2) + dz^2 \right] \\ &= \frac{R^2}{z^2} \left[e^{2hx'} (-dt'^2 + dx'^2) + dz^2 \right]. \end{aligned}$$

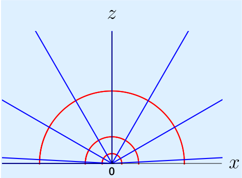
- There are two asymptotic boundaries (hence two CFTs), which are entangled.

- (The entanglement Hamiltonian of the finite interval can be discussed similarly.

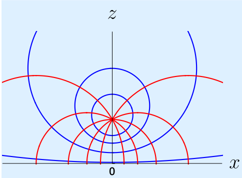
Different foliations $\leftrightarrow$ Different time-evolution



Flat



Rainbow



Möbius (SSD)

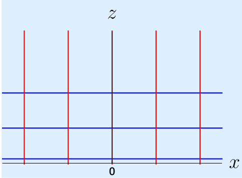
Bulk metric:

$$ds^2_{AdS_3} = R^2 \frac{dz^2 + dx^2 - dt^2}{z^2}$$

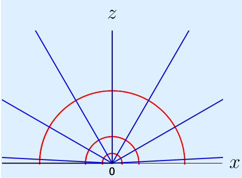
$$ds^2_{AdS_3} = \left[\frac{h^2 R^2}{\cos^2(h\Theta)}\right] \left[d\Theta^2 + \frac{1}{h^2\eta^2}\left(d\eta^2 - dt^2\right)\right]$$

$$ds^2_{AdS_3} = \frac{1}{\sinh^2 u}\left[du^2 + dv^2 - a^{-2}(\cosh u - \cos v)^2 dt^2\right]$$

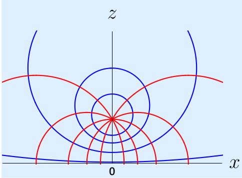
Different foliations $\leftrightarrow$ Different boundary metrics



Flat



Rainbow



Möbius (SSD)

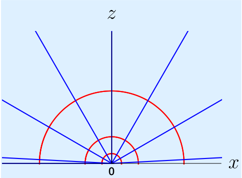
Slice metric:

$$ds^2_{Mink} = dx^2 - dt^2$$

$$ds^2_{AdS_2} = \frac{1}{h^2\eta^2}\left(d\eta^2 - dt^2\right)$$

$$ds^2_{Mobius} = -\left(1 - \tanh 2\gamma \cos \frac{2\pi x}{L}\right)^2 dt^2 + dx^2$$

Rainbow slicing



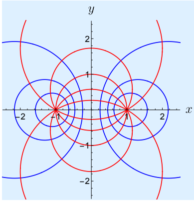
$$ds^2_{AdS_3} = \left[\frac{h^2 R^2}{\cos^2(h\Theta)}\right]\left[d\Theta^2 + \frac{1}{h^2\eta^2}\left(d\eta^2 - dt^2\right)\right]$$

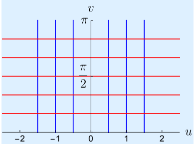
- There are two asymptotic boundaries (two CFTs) (similar to Rindler foliation)
- The two CFTs are connected at $(z, x) = 0$.
- Previously used, e.g., for AdS/BCFT [Takayanagi(11)]

C.f. Entanglement Hamiltonian and SSD in 2d CFT

- Conformal transformation:

$$w(z) = \log(z+R) - \log(z-R)$$





- EE hamiltonian on $[-R, +R] \to$ Hamiltonian with boundaries
- Transforming from strip to plane:

$$H = \int du \, T_{vv}|_{v_0=\pi} = \int_{-R}^{+R} dx \, \frac{(x^2 - R^2)}{2R} T_{yy}|_{y=0}$$

E.g., Casini-Huerta-Myers (11), Cardy-Tonni (16)

C.f. Entanglement Hamiltonian and SSD in 2d CFT

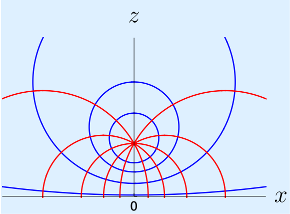
- Evolution in the "orthogonal direction" to the modular flow: $=$ SSD.
- Evolution operator:

$$H = \int_0^\pi dv\, T_{uu}(u_0, v) = r_0^2 \int_0^{2\pi} d\theta\, \frac{\cos\theta + \cosh u_0}{\sinh u_0}\, T_{rr}(r, \theta)$$

- In the limit $R \to 0$,

$$H \sim \int_0^L ds\, \sin^2\left(\frac{\pi s}{L}\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right)$$

Möbius foliation of AdS_3



- t-independent coord. transformation:

$$u + iv = \log(z + ix + a) - \log(z + ix - a) \overset{a \to 0}{\to} \frac{a}{z + ix}$$

- Metric:

$$ds^2_{AdS_3} = R^2 \frac{dz^2 + dx^2 - dt^2}{z^2} = \frac{R^2}{\sinh^2 u} \left[du^2 + dv^2 - a^{-2} (\cosh u - \cos v)^2 dt^2 \right]$$
$$\to \frac{R^2}{u^2} \left[dv^2 - a^{-2} (u^2 + v^2) dt^2 + du^2 \right]$$

The slice metric agrees with the one identified in [Wen-Wu (18)]

Finite temperature

- Start from the finite T (holographic) EE:

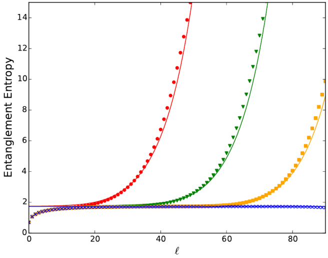
$$S_A(x_1, x_2; \beta) = \frac{c}{3} \log \left[\frac{\beta}{\pi \sqrt{\epsilon_1} \sqrt{\epsilon_2}} \sinh \left(\frac{\pi (x_2 - x_1)}{\beta} \right) \right].$$

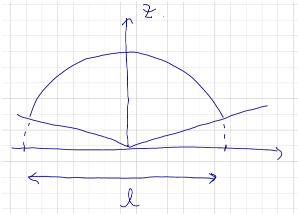
- Replace ϵ_1 and ϵ_2 with appropriate curvilinear cutoffs.

$$S_A(v_1, v_2; \beta) = \frac{c}{3} \log \left[\frac{\beta}{\pi \epsilon} \frac{\sinh \left(\frac{\pi}{\beta} (x(u=0, v_2) - x(u=0, v_1)) \right)}{\sqrt{\frac{\partial z(u=0, v_1)}{\partial u} \frac{\partial z(u=0, v_2)}{\partial u}}} \right].$$

- Good approximation when (length of interval) $\ll T$.

EE for Rainbow chain

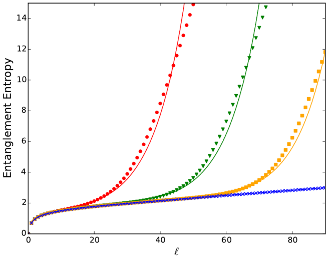


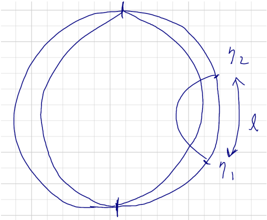


- "Defect" entanglement entropy

$$S_A(x;\beta,\epsilon) = \frac{c}{3}\log\left[\frac{\beta}{\pi h e^{h\ell}}\sinh\left(\frac{2\pi\epsilon e^{h\ell}}{\beta}\right)\right] + \cdots\,,$$

EE for Rainbow chain

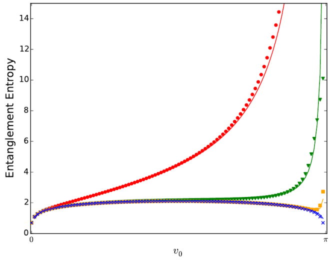


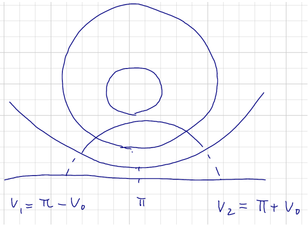


- "Half-chain" entanglement entropy

$$S_A(\ell;\beta,\eta_1) = \frac{c}{3}\log\left[\frac{\beta}{\epsilon\pi h\eta_1 e^{h\ell/2}}\sinh\left(\frac{\pi\eta_1(e^{h\ell}-1)}{\beta}\right)\right].$$

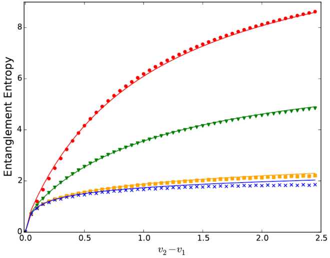
EE for Möbius evolution

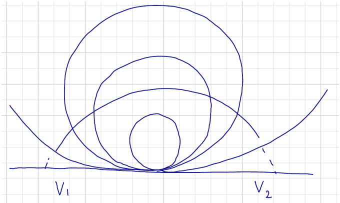




$$S_A(\pi - v_0, \pi + v_0; \beta) = \frac{c}{3} \log \left[\frac{4\beta}{L\epsilon} \cos^2 \left(\frac{v_0}{2} \right) \sinh \left(\frac{L}{\beta} \tan \left(\frac{v_0}{2} \right) \right) \right].$$

EE for SSD evolution





$$S_A = \frac{c}{3} \log \left[\frac{\beta |v_1 v_2|}{2\pi a \epsilon} \sinh \left(\frac{2\pi a}{\beta} \left| \frac{v_2 - v_1}{v_1 v_2} \right| \right) \right].$$

Summary and outlook

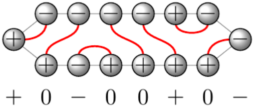
- Construction of holographic duals of rainbow chain and SSD.
- Computation of finite T entanglement entropy

Issues:

- Other inhomogenous systems?
- Other quantities, time-dependent setup ...
  Negativity and local quench [MacCormack-Kudler-Flam-SR]
- Higher dimensions ?

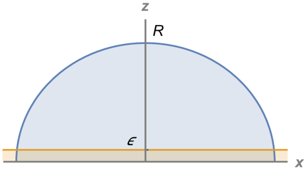
Rainbow chain, SPT, BCFT

- Folding rainbow chain → SPT phase [Nadir Samos Sáenz de Buruaga et al(18)]



- SPT phases ↔ BCFT [Qi-Katsura-Ludwig (11), Cho-Shiozaki-SR-Ludwig (16)]

- Rainbow foliation is closely related to BCFT.

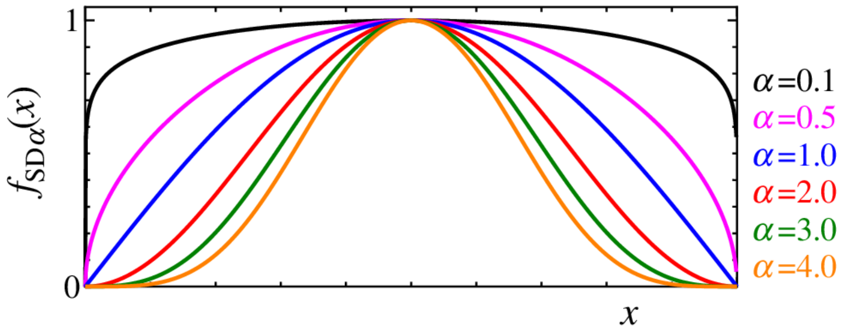


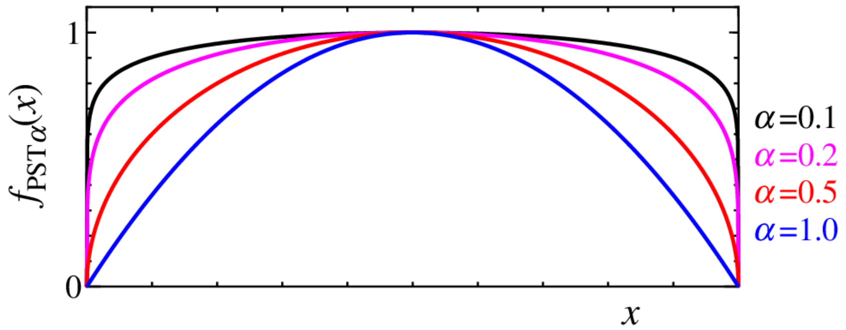
[Picture: Cavalcanti et al.(18)]

様々なエネルギースケール変換による一次元量子系の特性変化

引原 俊哉（群馬大）

Acknowledgement: 桂法称（東大）
上田宏（理研）
鈴木 隆史（兵県大）





11 Jul. 2019 @ SSD workshop, Riken

Outline

1. Introduction

   energy-scale deformation

   SSD

2. Sine-α deformation

   Long-distance entanglement

   Ground state properties of SDα chain

      local energy density, entanglement entropy

3. PST-α deformation

   Deformation for Perfect-State Transfer

   Ground state properties of PSTα chain

      local energy density, entanglement entropy

◆ Energy-scale deformation

$$\mathcal{H} = \sum_x f(x) h(x)$$

$f(x)$: scale function $\quad$ $h(x)$: hamiltonian density

We will consider the spin-1/2 XXZ chain

$$h(x_j) = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z$$

$\Delta = 0$: XX chain, free fermion

$\Delta \neq 0$: interacting

◆ Uniform XXZ chain

$$\mathcal{H} = \sum_j h(x_j) = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

Ground state (low-energy) properties
of the spin-1/2 XXZ chain in critical region
are well known
for thermodynamic limit / periodic b.c. / open b.c.

e.g.) for the uniform chain under open b.c.

$$\langle S_j^x S_k^x \rangle = A^x (-1)^{j-k} \frac{[g(j)g(k)]^{\eta/2}}{[g(\frac{j-k}{2})g(\frac{j+k}{2})]^{\eta}} + \cdots$$

$$d^a(x_j) = \langle S_j^a S_{j+1}^a \rangle = C_0^a + \frac{(-1)^j C_1^a}{[g(x_j)]^{1/2\eta}} + \cdots$$

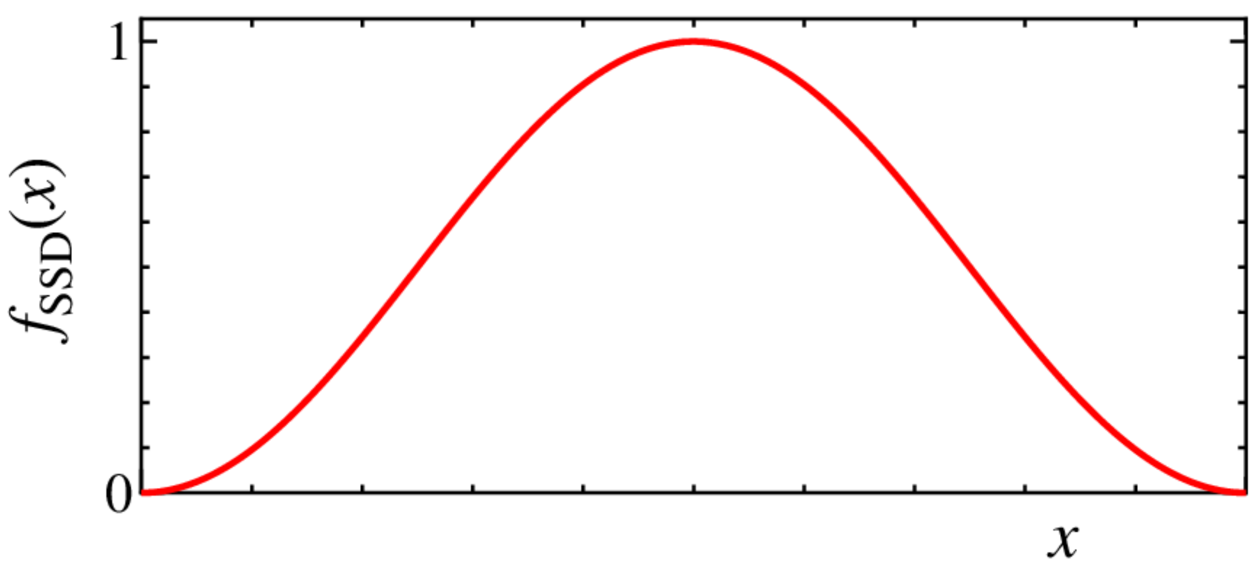
$$g(x_j) = \frac{L+1}{\pi} \sin\left(\frac{\pi x_j}{L+1}\right)$$

SSD

Sine-Square Deformation (SSD) Gendiar et al. (2009)

$$\mathcal{H}_{\mathrm{SSD}} = \sum_{x} f_{\mathrm{SSD}}(x) h(x)$$

$$f_{\mathrm{SSD}}(x) = \sin^2\left(\frac{\pi(x-\frac{1}{2})}{L}\right)$$



◆ SSD for 1d critical system

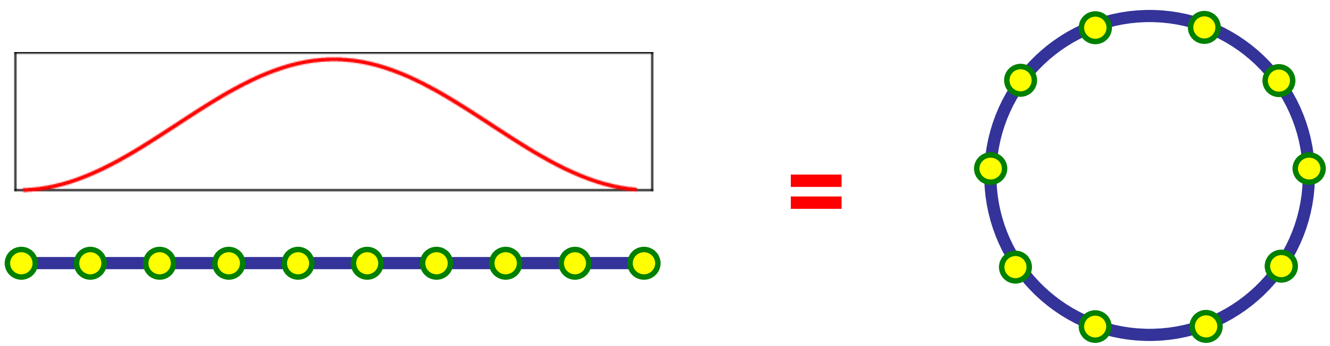
1d critical (gapless) systems under SSD

ground state is equivalent with that of periodic system

Hikihara-Nishino (2012)

Katsura (2011), Maruyama et al. (2011)

Hotta-Shibata(2011)



conformal mapping to infinite uniform chain

Wen-Ryu-Ludwig (2016)

grand-canonical analysis for magnetization curve

Hotta-Shibata(2012), Nishimoto et al.(2013)

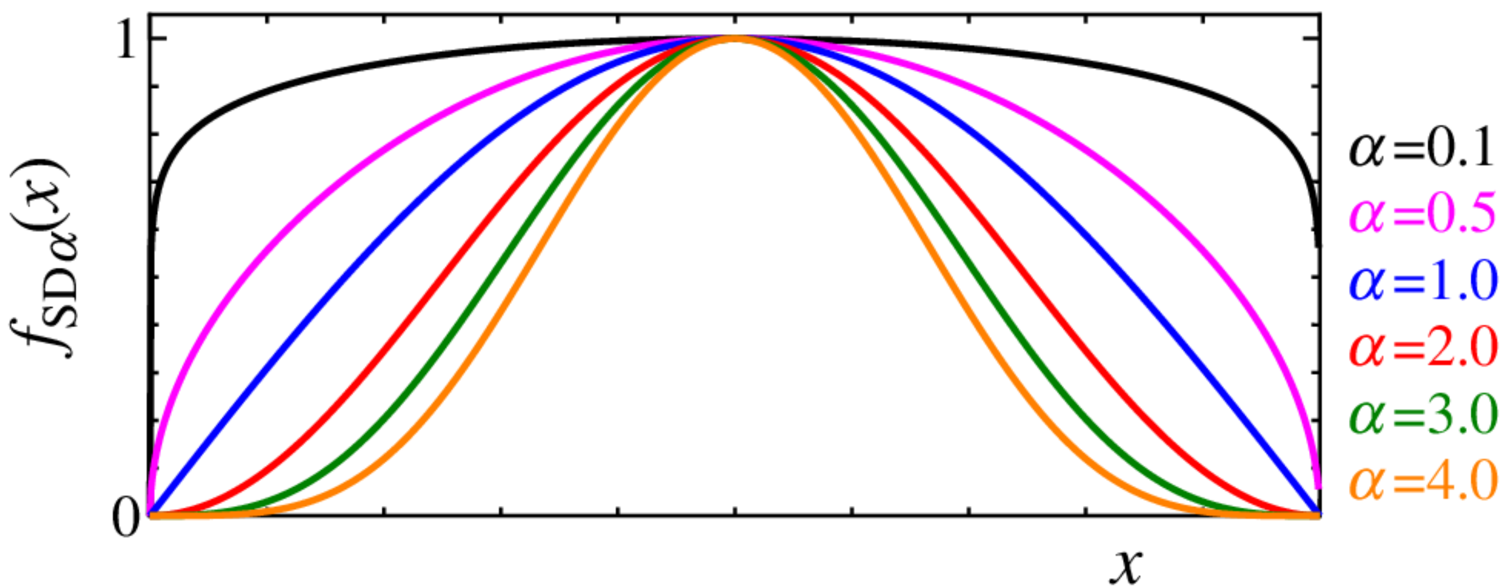
Sine-α deformation

Sine-α deformation (SDα)

Gendiar et al. (2009)

$$\mathcal{H}_{\mathrm{SD}\alpha} = \sum_x f_{\mathrm{SD}\alpha}(x) h(x)$$

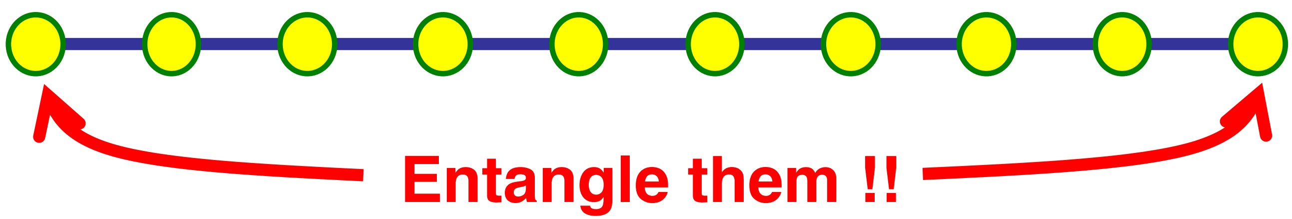
$$f_{\mathrm{SD}\alpha}(x) = \sin^\alpha\left(\frac{\pi\left(x-\frac{1}{2}\right)}{L}\right)$$



Long Distance Entanglement

For realizing Quantum-Information process,
genelation of large entanglement
between qubits located at a large distance
and connected by steady channel
is desirable

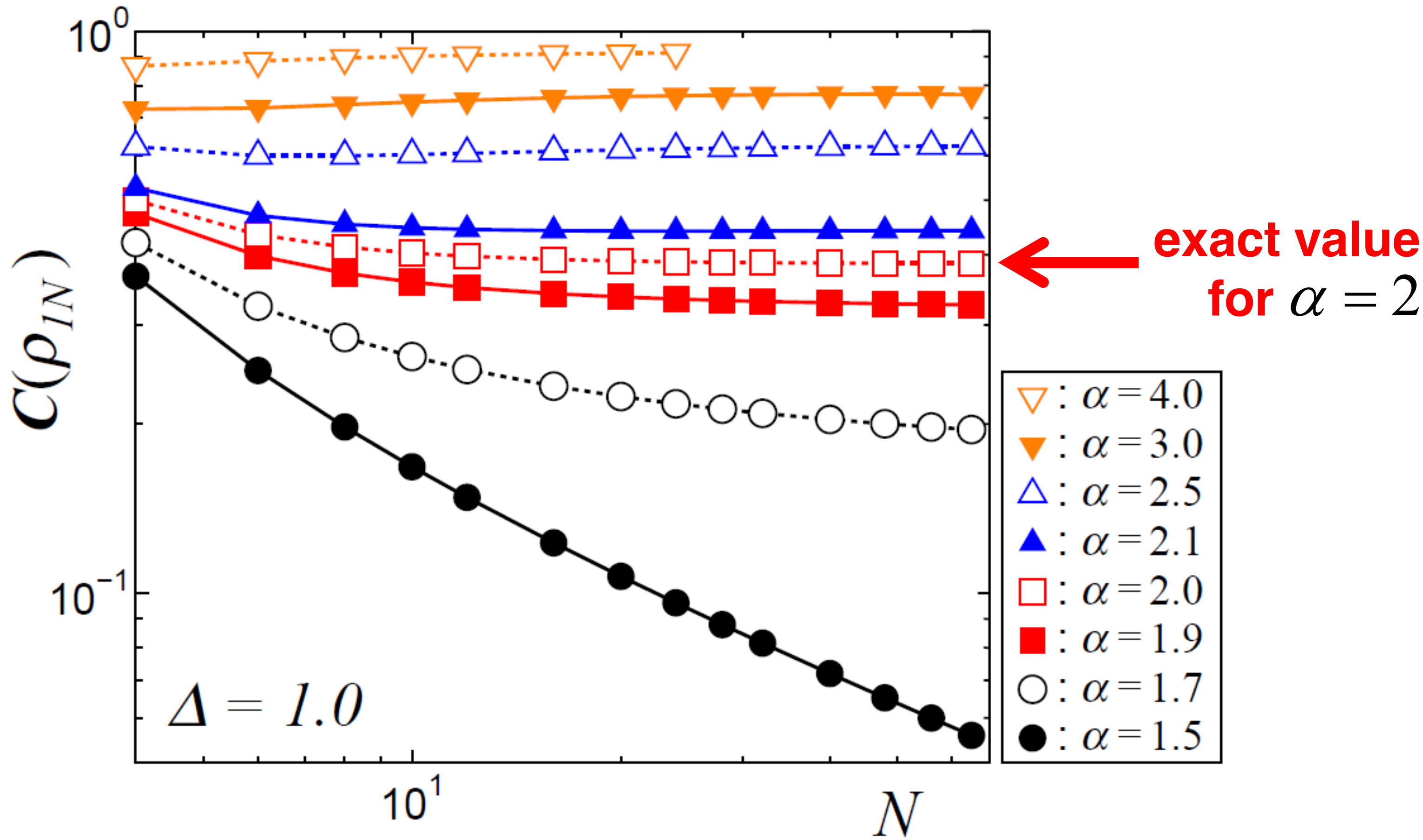
End-to-end entanglement in 1D quantum systems



SSD system realize true Long-Distance Entanglement

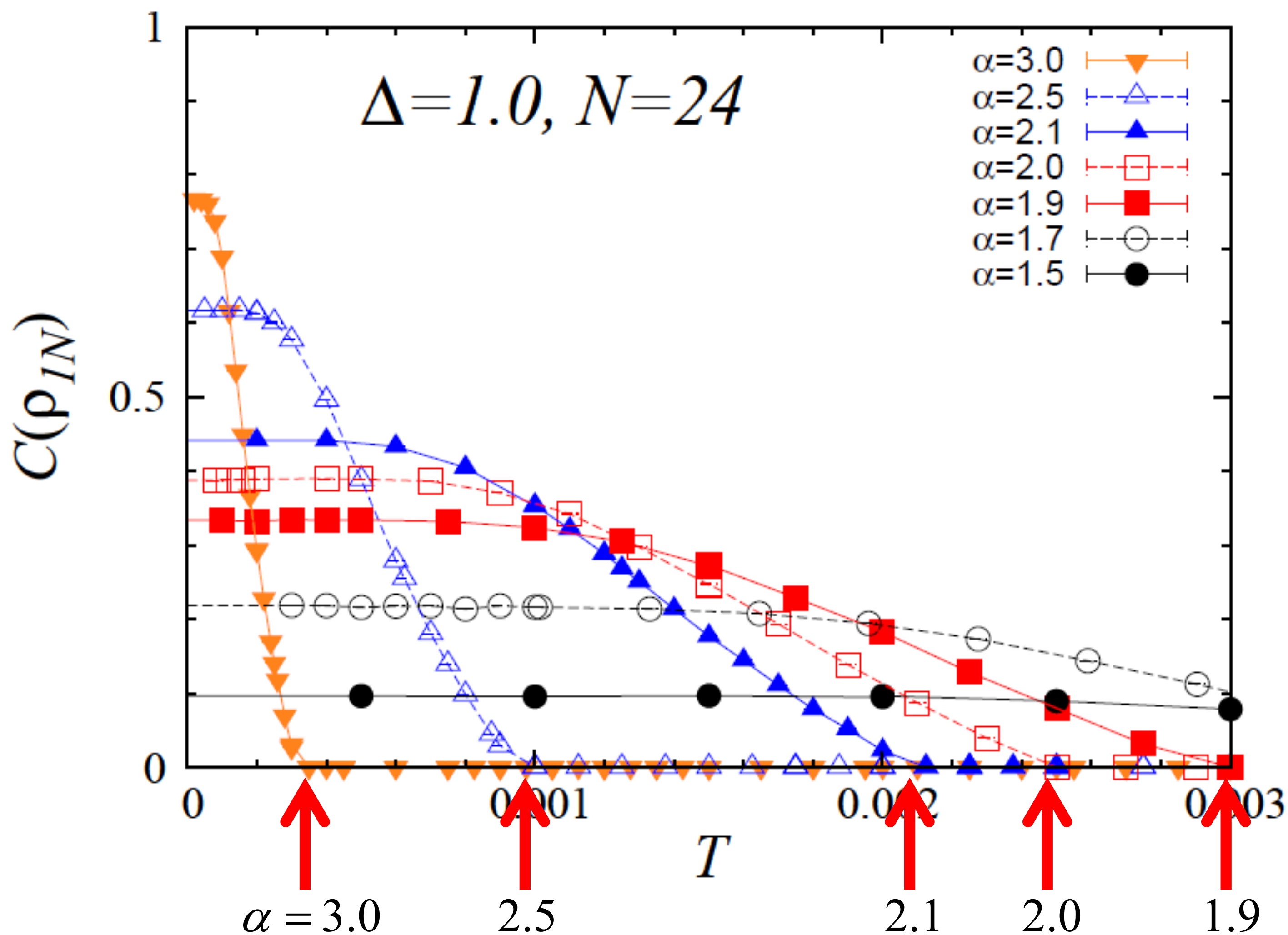
How end-to-end entanglement in SDα chain
depends on system size, temperature, …

End-to-end entanglement at T=0



- End-to-end concurrence for $\alpha \geq 2$ converges a finite value at $N \to \infty$ **: true LDE**
- Concurrence is larger as α is larger (coupling constants around edges are smaller)

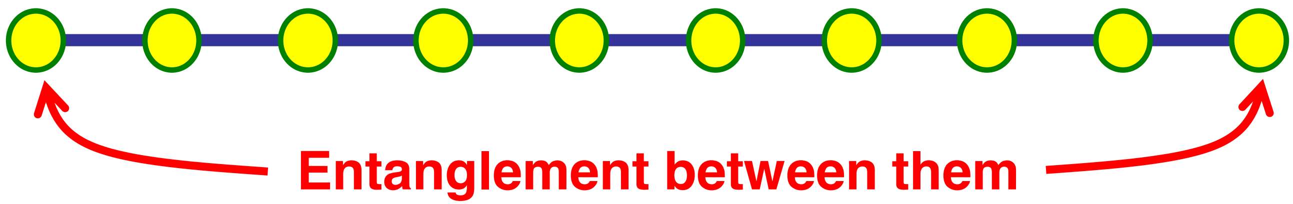
End-to-end entanglement at finite Temperatures



- Critical temperature T^* is smaller as α is larger (coupling constants around edges are smaller)

◆ Long Distance Entanglement

End-to-end entanglement in 1D quantum systems



$$\mathcal{H}_{\mathrm{SD}\alpha} = \sum_x f_{\mathrm{SD}\alpha}(x) h(x) \qquad f_{\mathrm{SD}\alpha}(x) = \sin^\alpha\left(\frac{\pi(x-\frac{1}{2})}{L}\right)$$

true LDE realizes for $\alpha \geq 2$

As α is larger,

end-to-end entanglement at T=0 larger

critical temperature T* smaller (LDE becomes fragile)

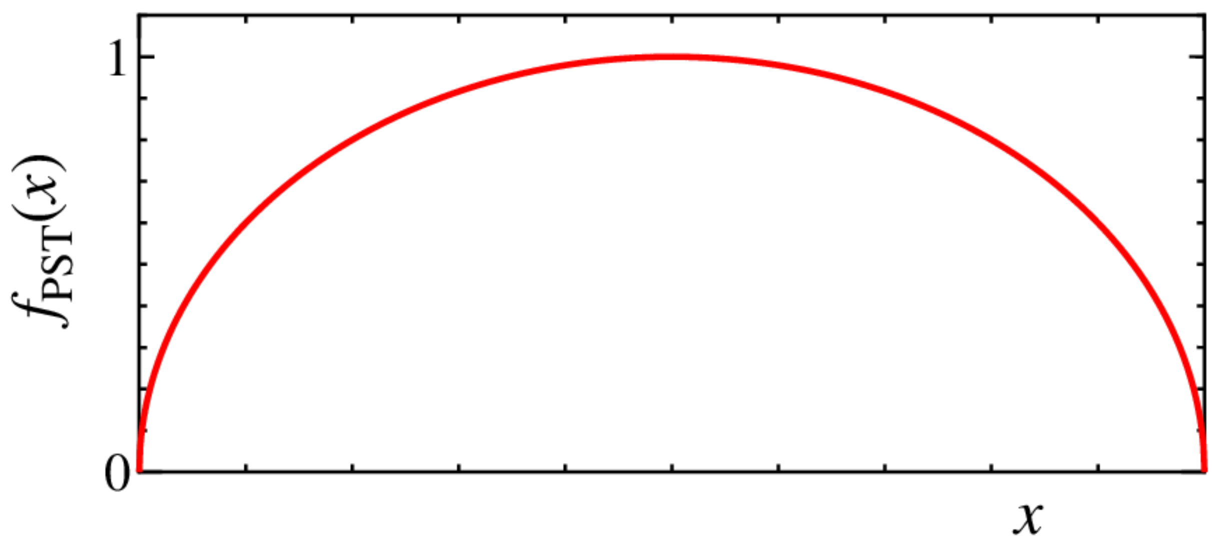
◆ Perfect-State Transfer

Energy-scale deformation for Perfect-State Transfer

Christandl et al. (2004)

$$\mathcal{H}_{\mathrm{PST}} = \sum_x f_{\mathrm{PST}}(x) h(x)$$

$$f_{\mathrm{PST}}(x) = \sqrt{\left(x - \frac{1}{2}\right)\left(L - x + \frac{1}{2}\right)}$$



◆ Perfect-State Transfer

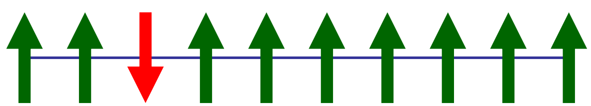
Perfect-State Transfer in XX-spin chain

Christandl et al. (2004)

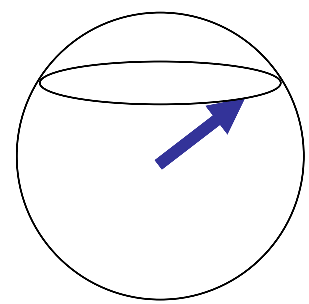
$$\mathcal{H}_{\mathrm{PST}} = \sum_{j=1}^{L-1} \sqrt{j(L-j)}\left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right)$$

L-site system in one-magnon subspace

Single $S = (L\text{-}1)/2$ spin in transverse field



$$|j\rangle = S_j^- |\mathrm{all} \uparrow\rangle$$



$$|S^z = j\rangle$$

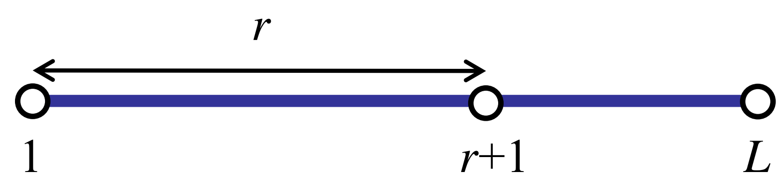
hopping amplitude

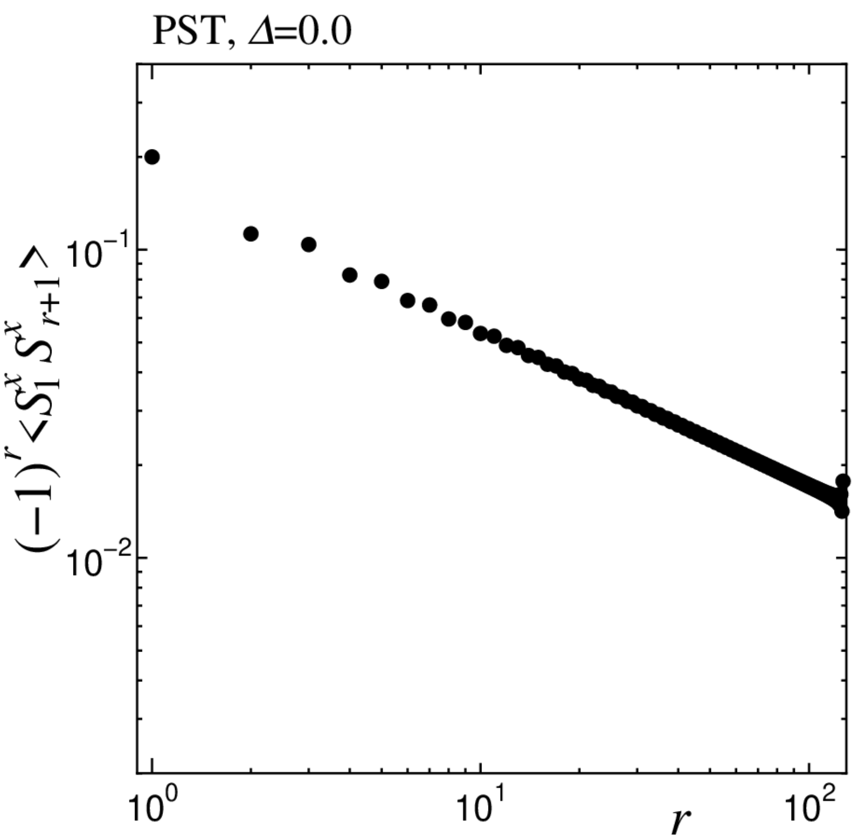
Clebsch-Gordan coeff.

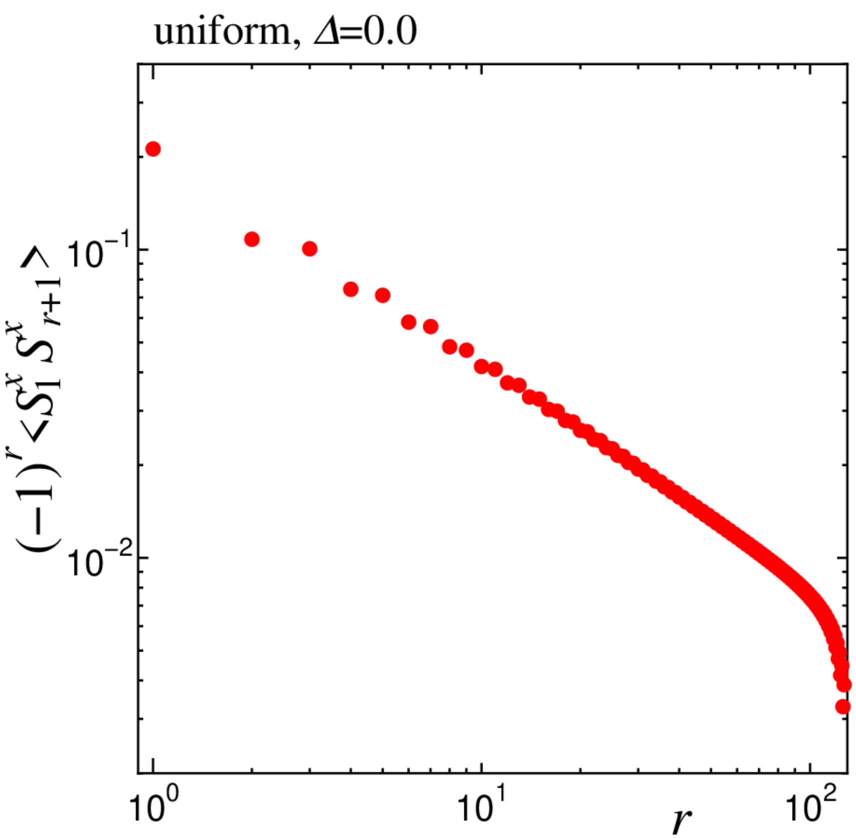
Quantum-state transfer without loss of fidelity

Constant level spacing of eigenenergies

◆ Transverse spin correlation in PST system







◆ CFT from open chain to PST system

Inverse sine mapping Wen-Ryu-Ludwig (2016)

$$z = R \sin \omega$$

ω plane		z plane
$-\frac{\pi}{2} < u < \frac{\pi}{2}, \quad v = 0$	$\leftrightarrow$	$-R < x < R, \quad y = 0$
uniform chain		PST chain

leading term in staggered part of $\langle S_j^x S_k^x \rangle$

$$\frac{A^x (-1)^{j-k} \left[g(u_1) g(u_2) \right]^{\eta/2}}{\left[g(\frac{u_1 - u_2}{2}) g(\frac{u_1 + u_2}{2}) \right]^{\eta}} \quad \leftrightarrow \quad \frac{A^x (-1)^{j-k}}{|x_1 - x_2|^{\eta}}$$

Virasoro algebra in K-space

Syoji Zeze

Yokote Seiryo Gakuin High School

arXiv:1906.03576

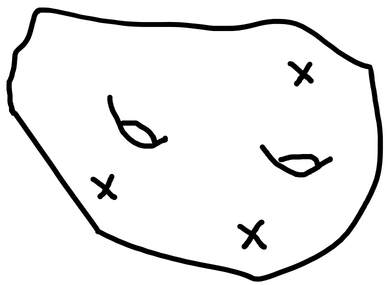
11 July, 2019 @RIKEN SSD2019

Summary

- **SSD-based (inspired) derivation** of novel representation of Virasoro algebra
- **wedge-based framework of open string field theory**
- 0, 1, 2 D-branes

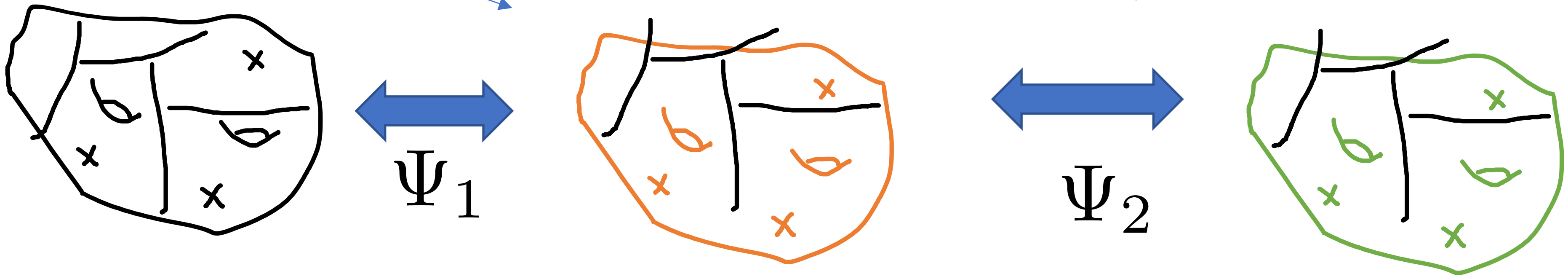
$$\mathbb{L}_m = -K^m L = -K^{m+1}\partial_K$$

String theory = CFT on Reimann surfaces



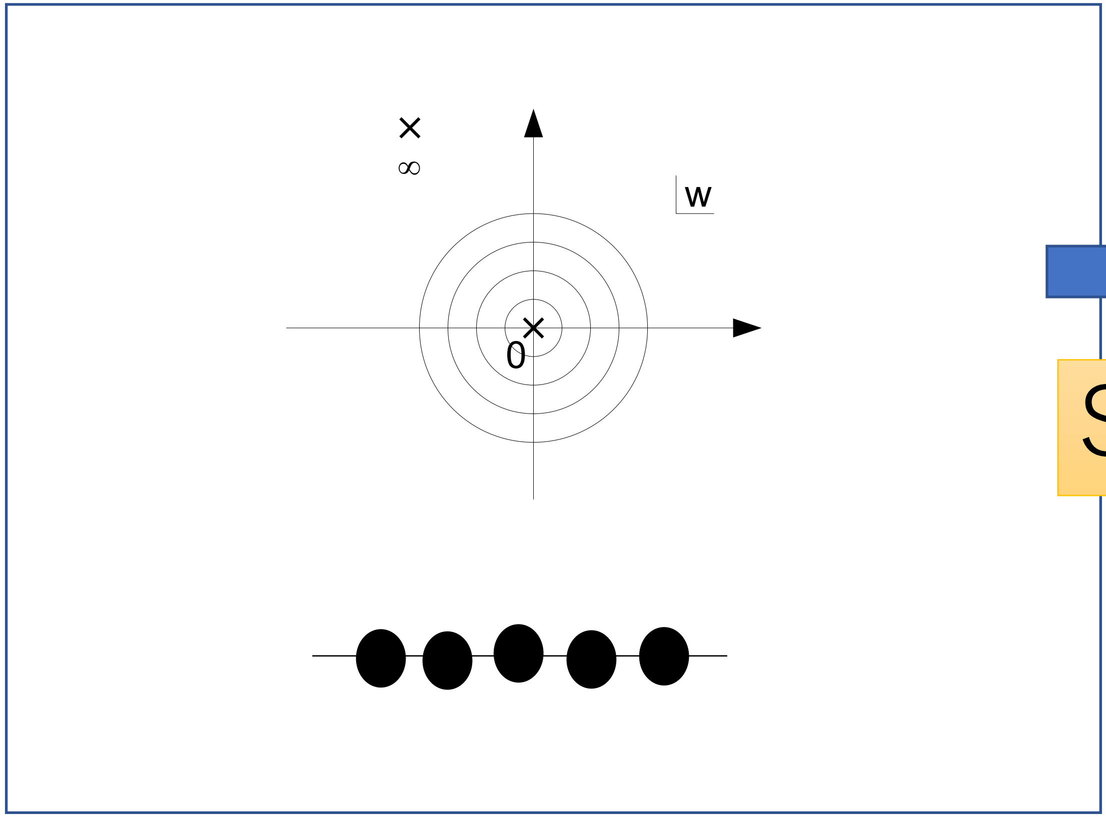
String field theory = CFTs (and more?)

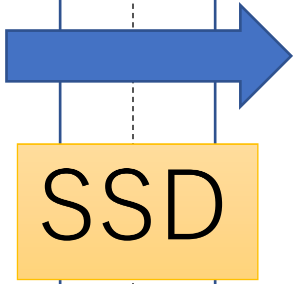
$$S \sim \int \left(\frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi^3 \right)$$

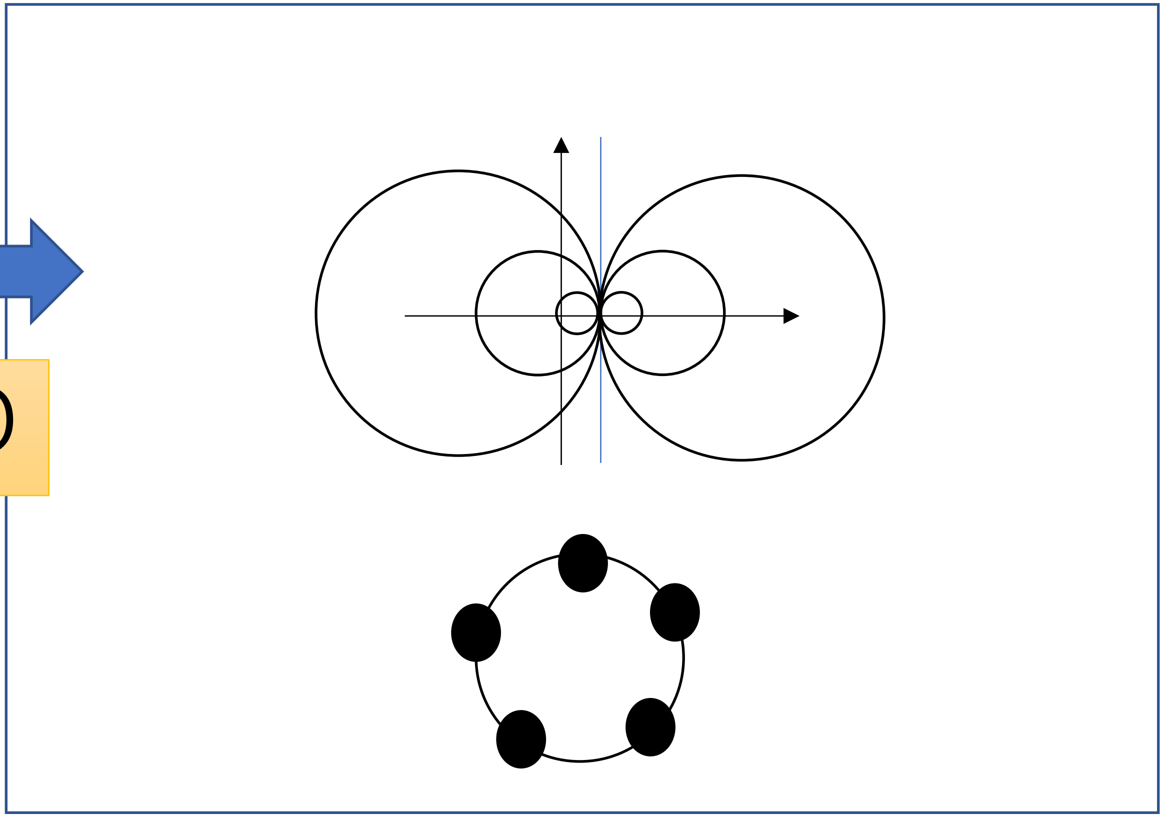


SSD as CFT: Dipolar quantization

Ishibashi-Tada







$$H = L_0$$

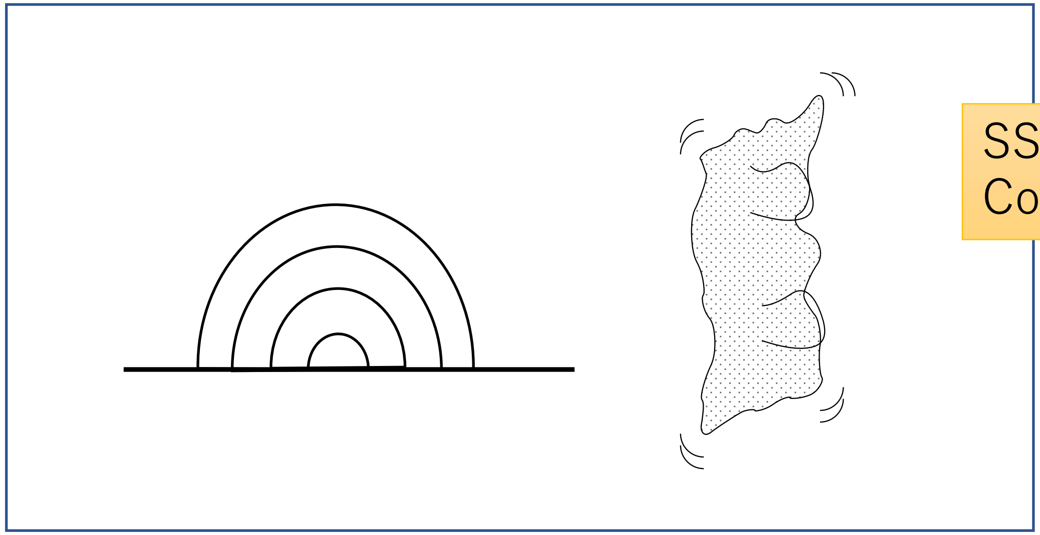
L_m :Discrete

$$H = L_0 - \frac{1}{2}(L_1 + L_{-1})$$

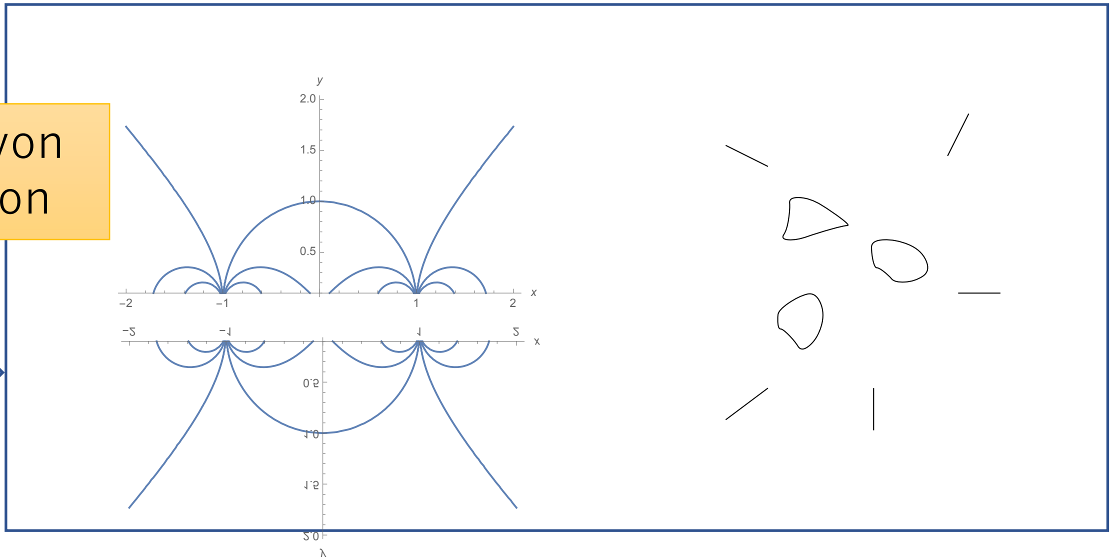
$\mathcal{L}_\kappa$:Continuous

SSD as open string field theory

Kishimoto-Kitade-Takahashi
S. Z.



SSD=Tachyon Condensation



$$H \sim L_0$$

L_m :Discrete

$$H \sim L'_0 - \frac{1}{2}\left(L'_2 + L'_{-2}\right) + const.$$

$\mathcal{L}_\kappa$:Continuous

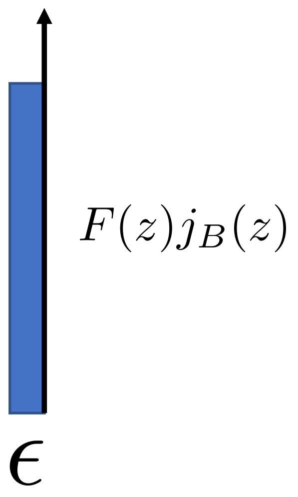
Closed string

Two frameworks for OSFT solutions

Identity based Takahashi-Tanimoto

$$\Psi = \left(\int_C dz F(z) j_B(z)\right) I + \cdots$$

$$\boxed{F(z)} = -\frac{1}{2} - \frac{1}{4}\left(z^2 + z^{-2}\right)$$



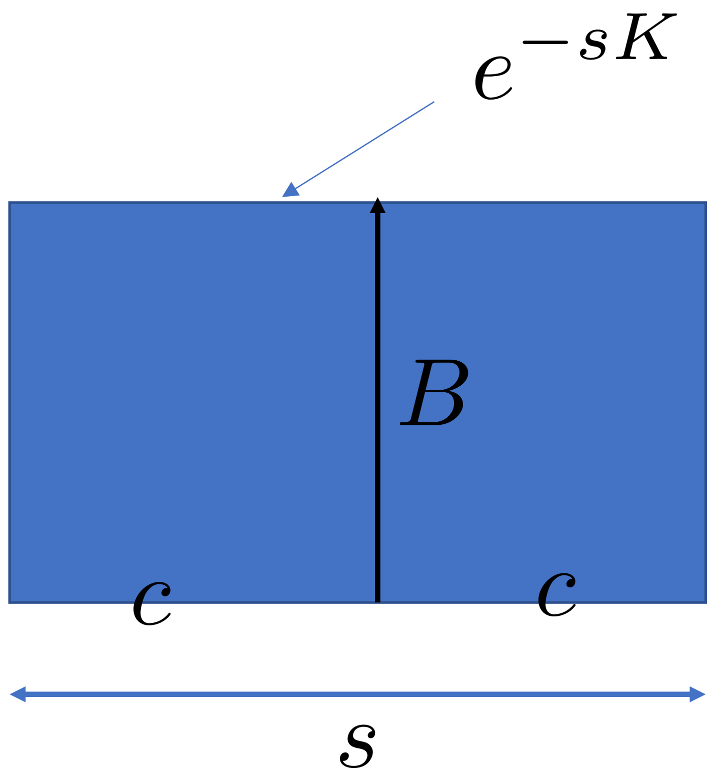
SSD!

Wedge based Schnabl

$$\Psi_G = \sqrt{1-G}\, c\frac{K}{G} Bc\sqrt{1-G}$$

$$\boxed{G(K)}$$

$$\int_0^\infty ds g(s)$$



SSD ?

Question

How SSD will work in
wedge-based framework ?

Answer

Virasoro algebra in K-space

KBcL algebra

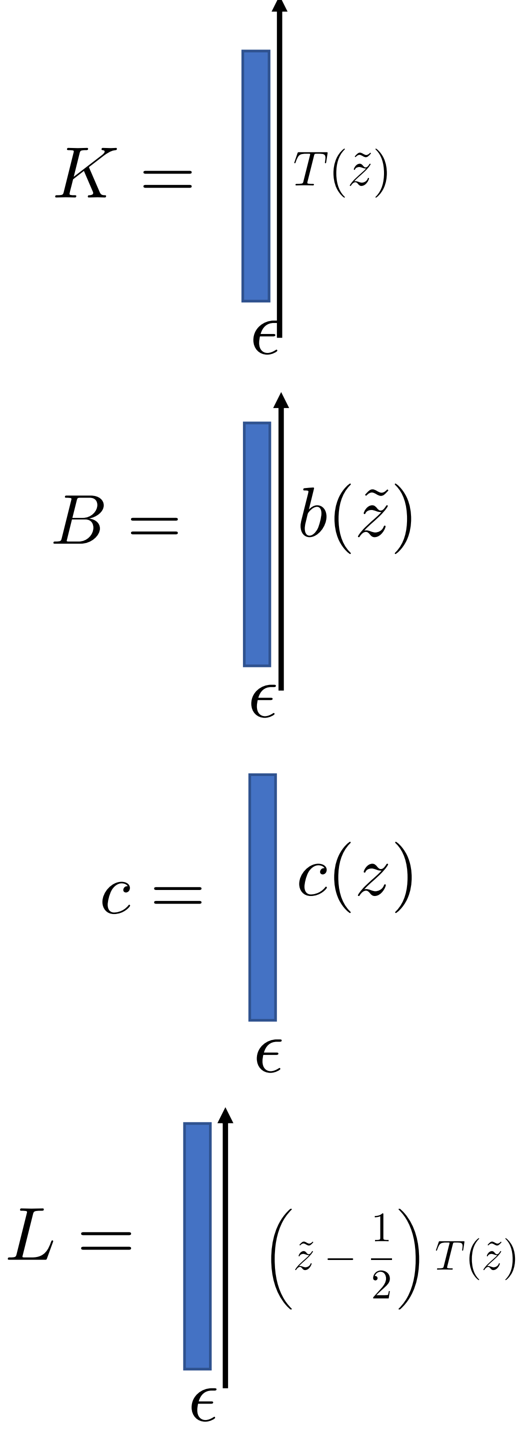
Mertes-Schnabl

$$Q_B c = cKc, \quad Q_B K = 0, \quad Q_B B = K, \quad \boxed{Q_B L = 0,}$$

$$cB + Bc = 1, \quad c^2 = 0,$$

$$\boxed{\begin{array}{l}[L, K] = K, \quad [L, B] = B, \quad [L, c] = -c. \\ \\ [L, f(K)] = K\partial f(K)\end{array}}$$

$$\boxed{\begin{array}{l}\mathbb{L}_m = -K^m L = -K^{m+1}\partial_K \\ \\ [\mathbb{L}_m, \mathbb{L}_n] = (m-n)\mathbb{L}_{m+n}\end{array}}$$



Virasoro generators for nontrivial backgrounds (= SSD!)

$$\Psi_G = \sqrt{1-G}c\frac{K}{G}Bc\sqrt{1-G}$$

$$\hat{Q}\Psi = Q_B\Psi + \Psi_G\Psi + \Psi\Psi_G$$

$$u_\lambda = \phi^\lambda u_0,$$

$$v_\lambda = -\frac{\phi^\lambda}{G} - \phi^\lambda\frac{F}{G}BcF - FBc\frac{F}{G}\phi^\lambda$$

$$\boxed{\begin{aligned}\hat{\mathbb{L}}_\lambda &= \hat{Q}\hat{\mathbb{b}}_\lambda \\ &= u_\lambda + v_\lambda L,\end{aligned}}$$

$$\phi(K) = \exp\left(\int^K dK'\frac{G(K')}{K'}\right)$$

$$[\hat{\mathbb{L}}_\lambda, \hat{\mathbb{L}}_{\lambda'}] = (\lambda-\lambda')\hat{\mathbb{L}}_{\lambda+\lambda'}$$

0, 1, 2 branes

$$G(K) = \left(\frac{1+K}{K}\right)^n$$

$$\phi^\lambda(K) \sim \begin{cases} \boxed{(1+K)^\lambda} & \text{for the tachyon vacuum } (n=-1) \\ K^\lambda & \text{for perturbative vacuum } (n=0) \\ e^{-\frac{\lambda}{K}} K^\lambda & \text{for two-branes } (n=1) \end{cases}$$

Tachyon vacuum avoids K=0 singularity ! -> continuous ?

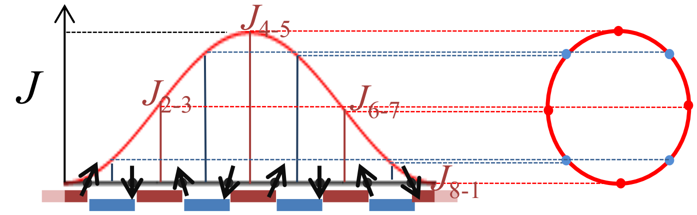
Remark

- More understanding of K-space required > K-space CFT ?
- Splitting (Closed strings)?

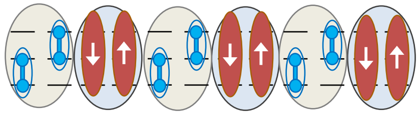
SSDを用いたフラストレート量子スピン系、多自由度電子系の解析

東北大理　柴田 尚和

- 並進対称性の破れと境界条件
- 2次元フラストレート量子スピン系
- 多自由度強相関電子系

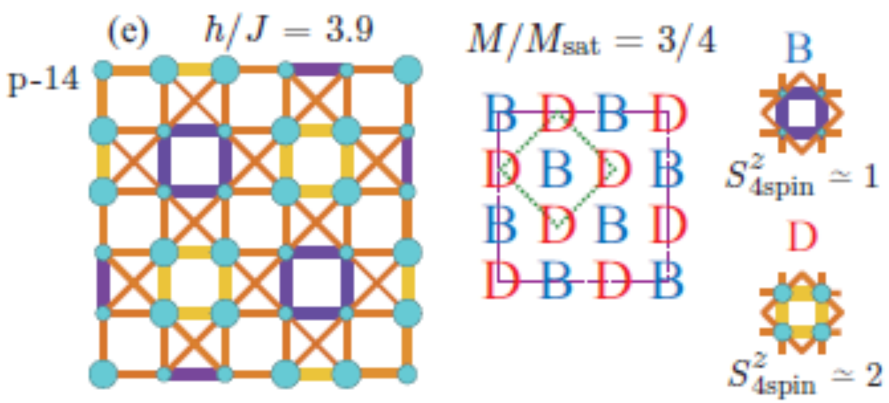


2-Channel Kondo Lattice



Kurebayashi and Shibata (2019)

Checkerboard Lattice

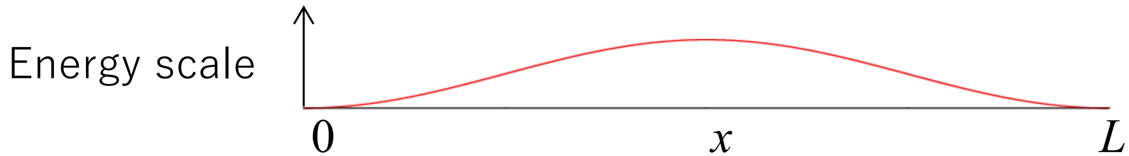


Morita and Shibata PRB (2016)

境界条件と対称性の破れ

$$f(x) = \sin^2\left(\frac{\pi}{L}x\right)$$

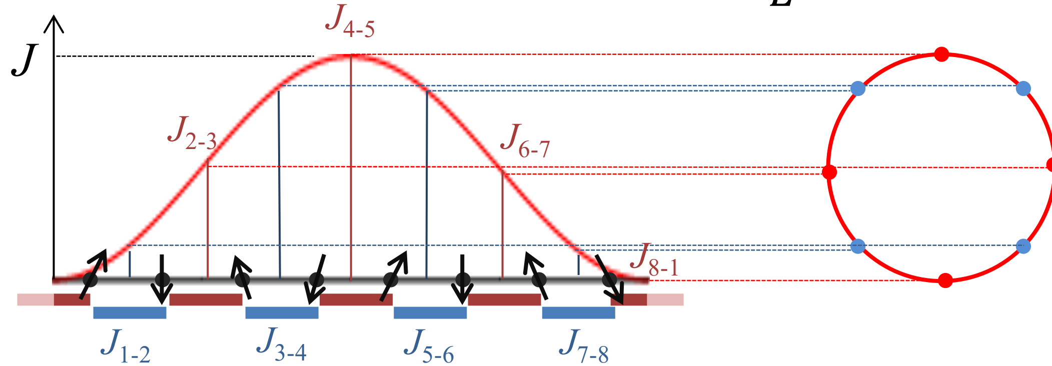
Deformation



A. Gendiar, R. Krcmar, and T. Nishino, Prog. Theor. Phys. **122**, 953 (2009); **123**, 393 (2010).

$$\sin^2\left(\frac{\pi}{L}x\right) = \frac{1-\cos\left(\frac{2\pi}{L}x\right)}{2}$$

$$J f(x) = J \sin^2\left(\frac{\pi}{L}x\right)$$

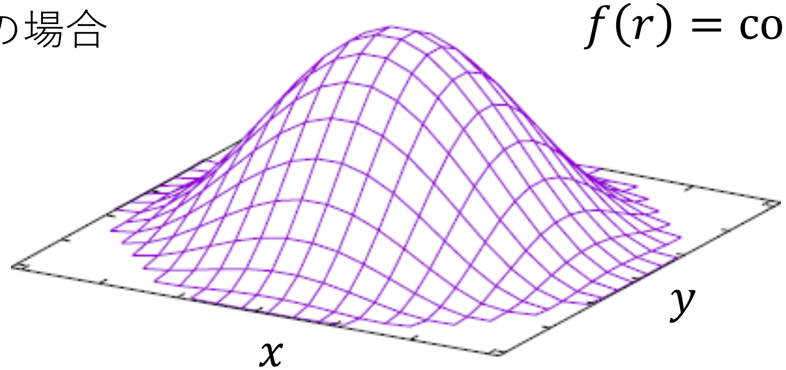


$$\boxed{J_{1\text{-}2}+J_{3\text{-}4}+J_{5\text{-}6}+J_{7\text{-}8} = J_{2\text{-}3}+J_{4\text{-}5}+J_{6\text{-}7}+J_{8\text{-}1} = JN/2}$$

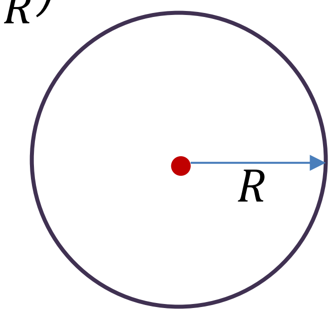
0

境界条件と対称性の破れ

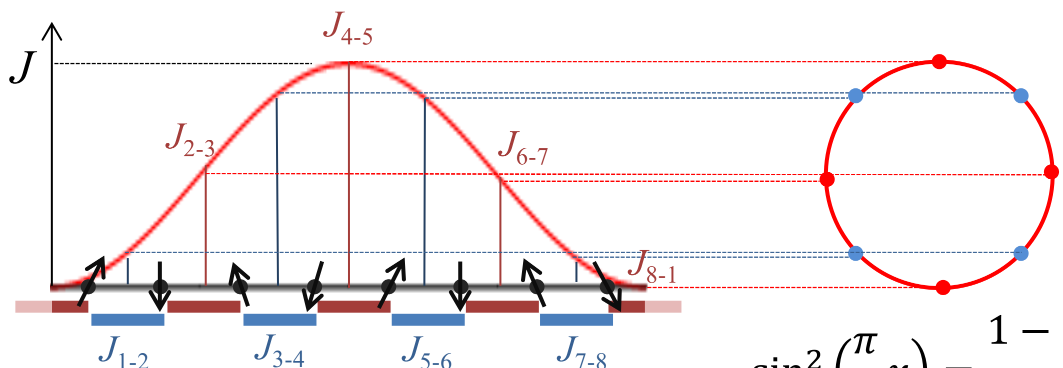
2次元の場合



$$f(r) = \cos^2\left(\frac{\pi r}{2R}\right)$$



$$J f(x) = J \sin^2\left(\frac{\pi}{L} x\right)$$



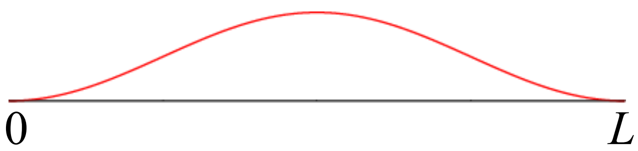
$$\sin^2\left(\frac{\pi}{L} x\right) = \frac{1 - \cos\left(\frac{2\pi}{L} x\right)}{2}$$

$$\boxed{J_{1\text{-}2} + J_{3\text{-}4} + J_{5\text{-}6} + J_{7\text{-}8} = J_{2\text{-}3} + J_{4\text{-}5} + J_{6\text{-}7} + \underset{\substack{\uparrow \\ 0}}{J_{8\text{-}1}} = JN/2}$$

Sine-Square Deformation

$$f(x) = \sin^2\left(\frac{\pi}{L}x\right)$$

$$\Delta k/2 = \frac{\pi}{L}$$



$$\Delta k = \frac{2\pi}{L}$$

$$
\begin{aligned}
f(x) &= \sin^2(\Delta k x/2) \\
&= \frac{1}{2}\{1-\cos(\Delta k x)\} \\
&= \frac{1}{2} - \frac{1}{4}\{\exp(i\Delta k x) + \exp(-i\Delta k x)\}
\end{aligned}
$$

$$H_{Original} = \sum_k (\varepsilon_k - \mu) c_k^+ c_k$$

$$
\begin{aligned}
H_{Deform} = & \sum_k \frac{1}{2}(\varepsilon_k - \mu)\, c_k^+ c_k \\
& - \sum_k \frac{1}{4} e^{-i\frac{\Delta k}{2}} (\varepsilon_{k+\frac{\Delta k}{2}} - \mu)\, c_{k+\Delta k}^+ c_k \\
& - \sum_k \frac{1}{4} e^{i\frac{\Delta k}{2}} (\varepsilon_{k-\frac{\Delta k}{2}} - \mu)\, c_{k-\Delta k}^+ c_k
\end{aligned}
$$

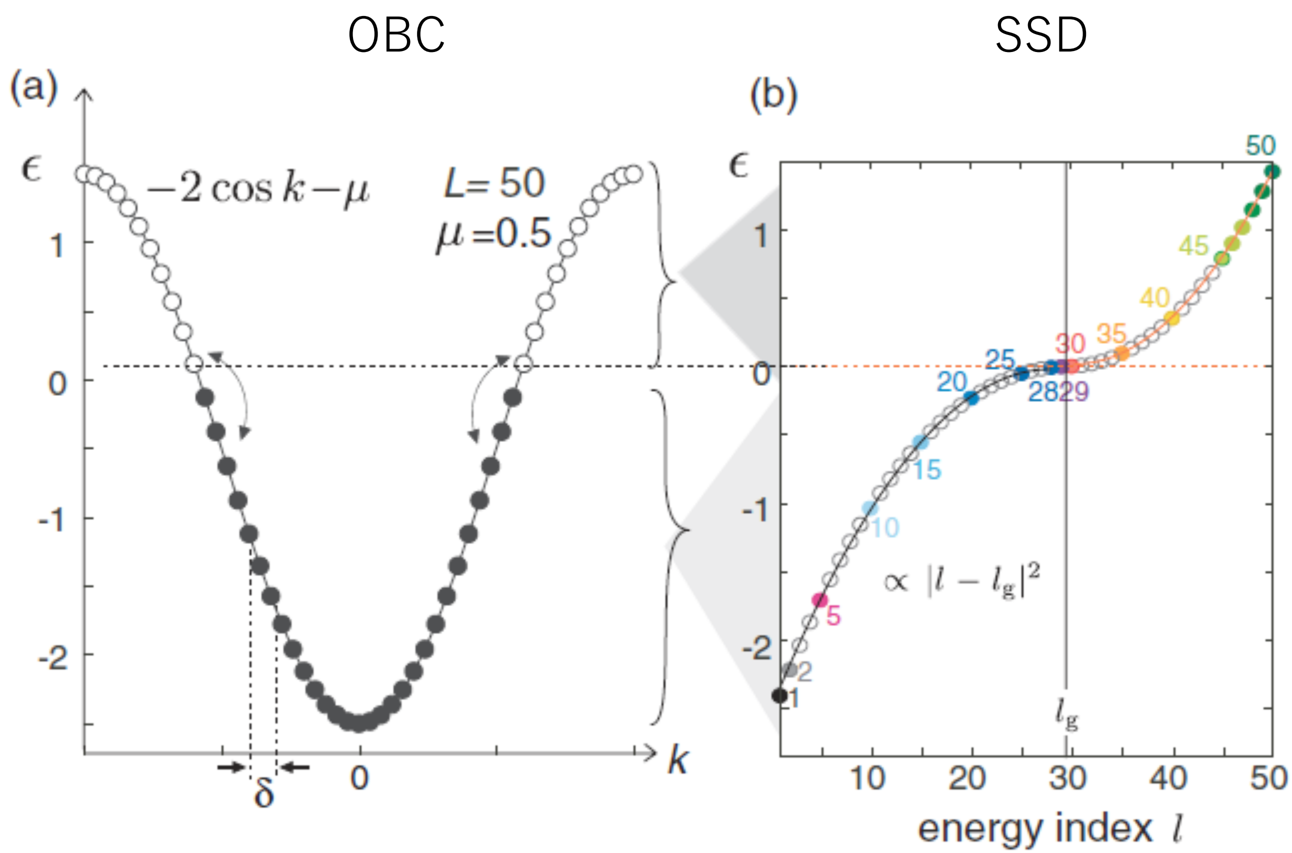
$$
H_{Deform} = \begin{bmatrix}
1 & -0.4 & & & & & & & & -0.4 \\
-0.4 & 0.7 & -0.3 & & & & & & & \\
 & -0.3 & 0.3 & 0 & & & & & & \\
 & & 0 & -0.3 & 0.3 & & & & & \\
 & & & 0.3 & -0.7 & 0.4 & & & & \\
 & & & & 0.4 & -1 & 0.4 & & & \\
 & & & & & 0.4 & -0.7 & 0.3 & & \\
 & & & & & & 0.3 & -0.3 & 0 & \\
 & & & & & & & 0 & 0.3 & -0.3 \\
-0.4 & & & & & & & & -0.3 & 0.7
\end{bmatrix}
$$

Positive and negative eigenvales are separated

Ground state many-body wavefunction is identical to the original one

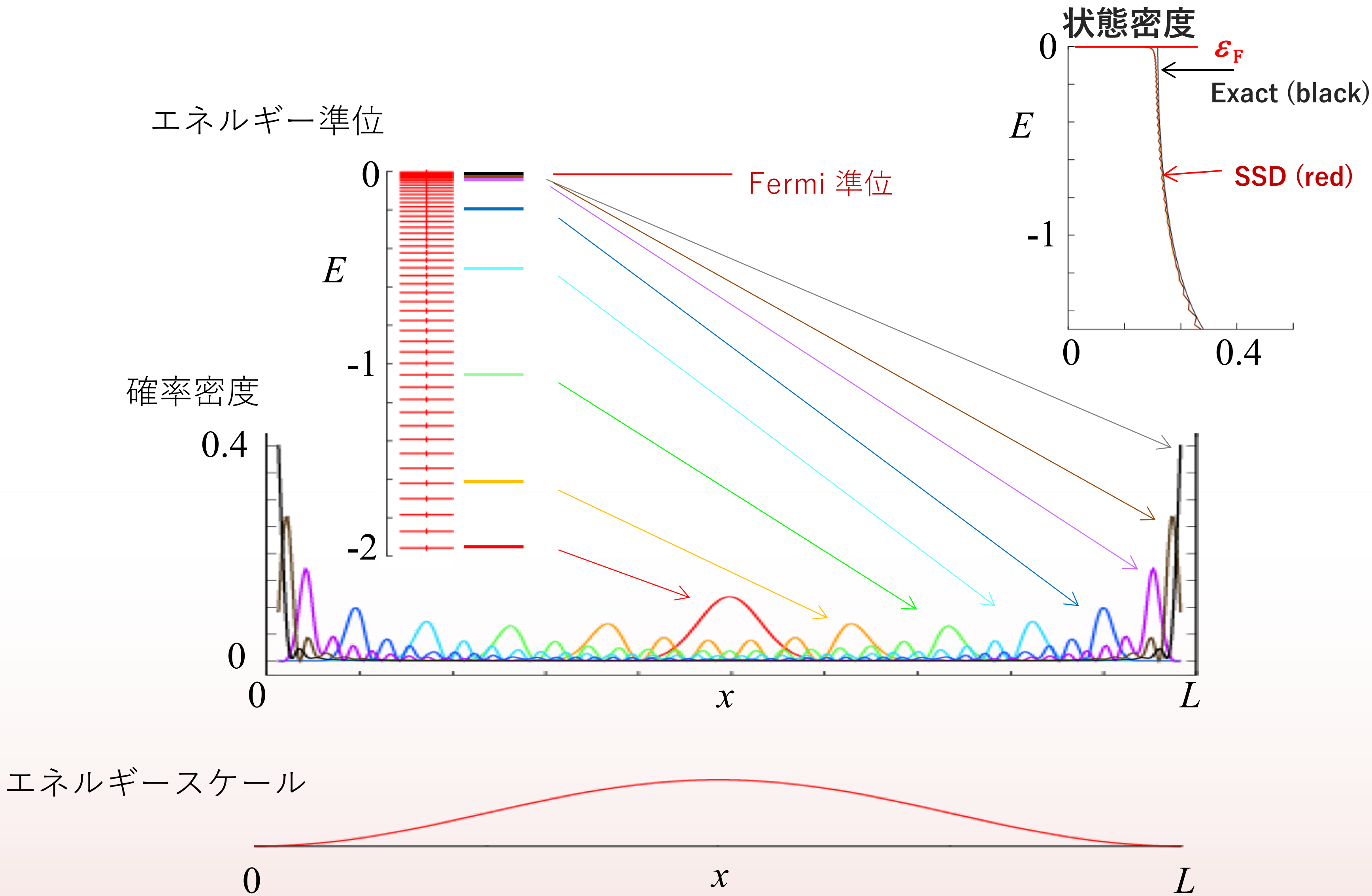
T. Hikihara and T. Nishino, Phys. Rev. B **83**, 060414(R) (2011)
H. Katsura, J. Phys. A **44**, 252001 (2011)

Energy spectrum



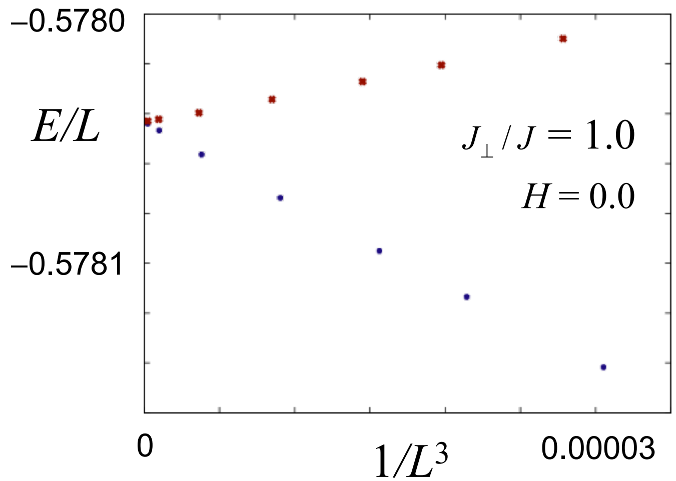
Hotta and Shibata PRB **86** 041108 (2012)

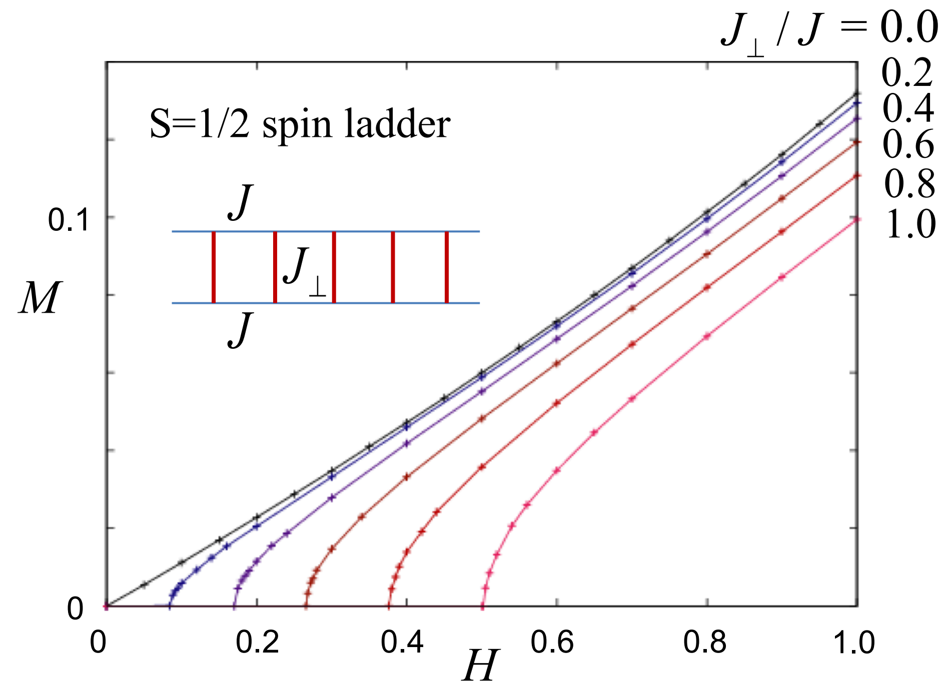
エネルギー準位と確率密度分布

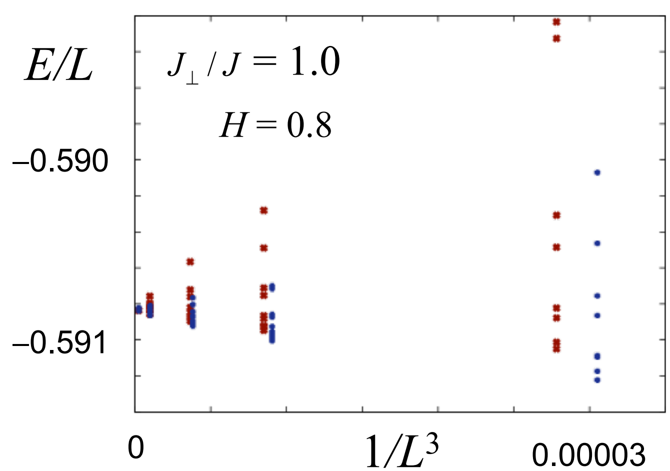


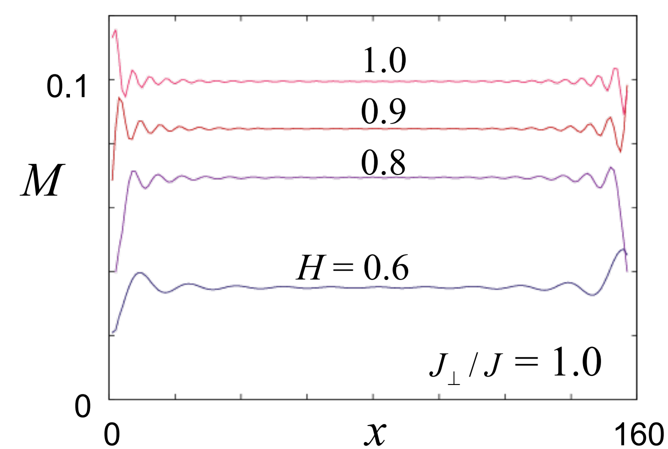
Size scaling and magnetization (SSD)

S=1/2 Heisenberg spin ladder

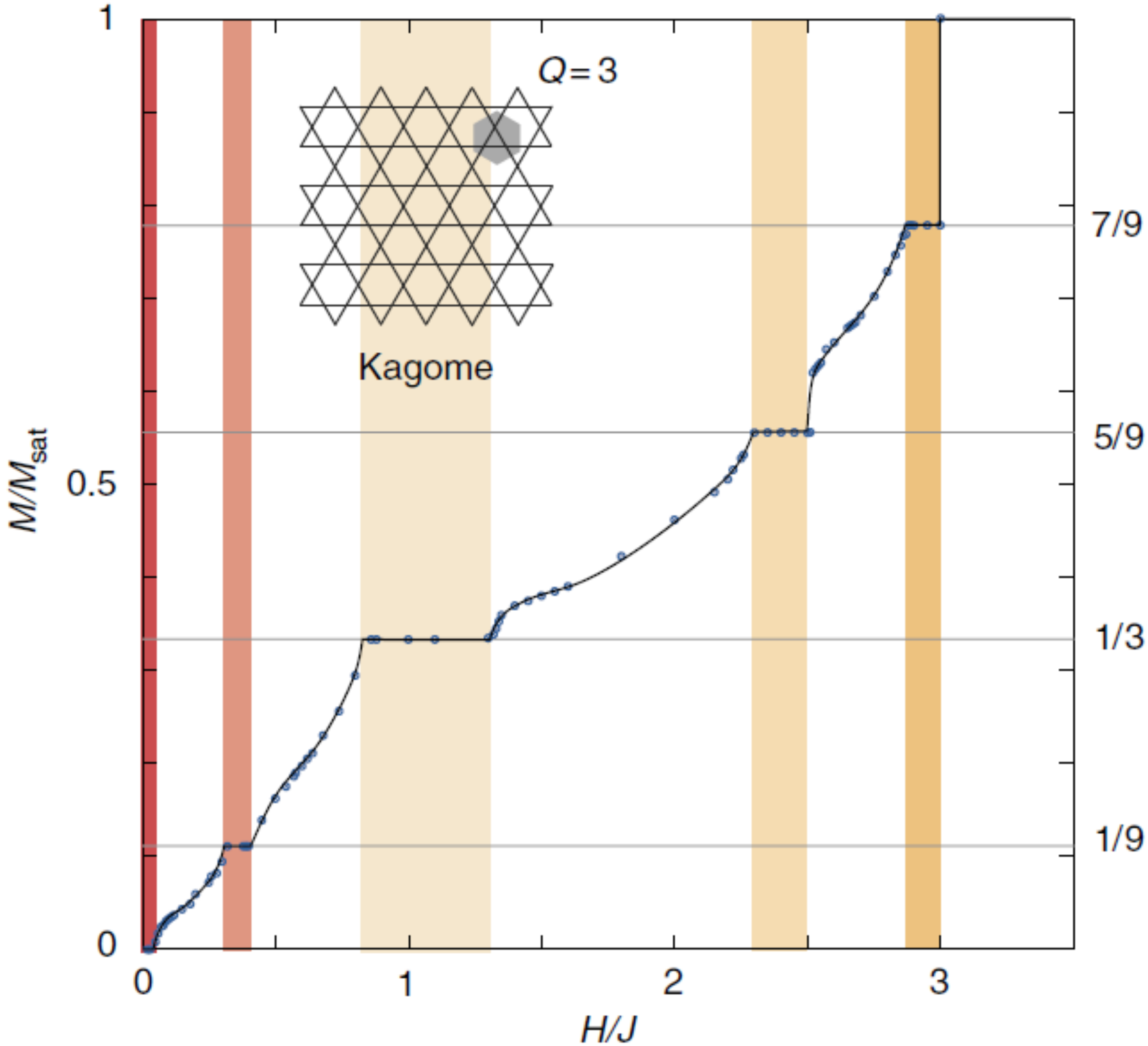


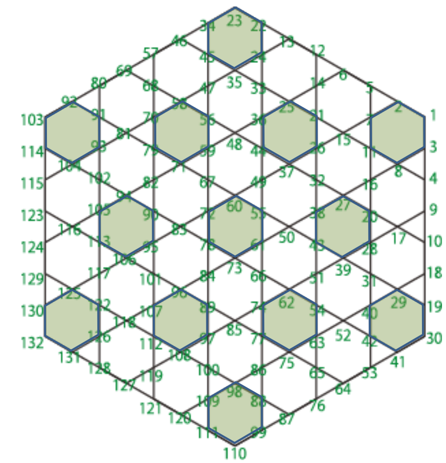


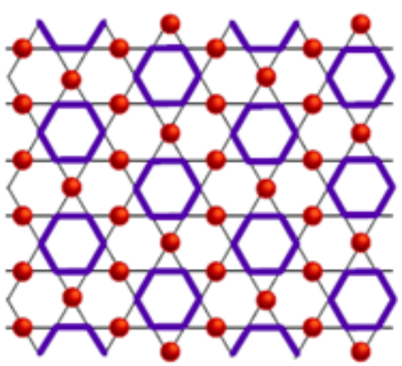




Kagome lattice



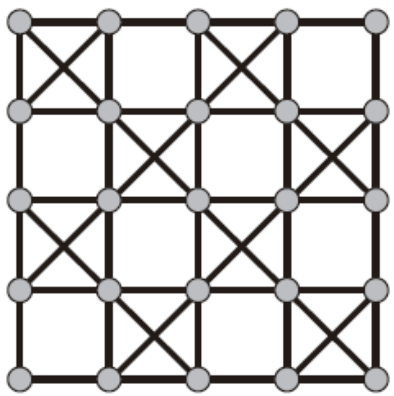




Nishimoto, Shibata and Hotta: Nature Communications **4** 2287 (2013)

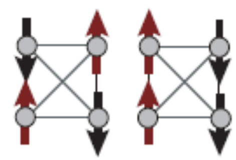
Checkerboard lattice

S=1/2 antiferromagnetic Heisenberg interaction

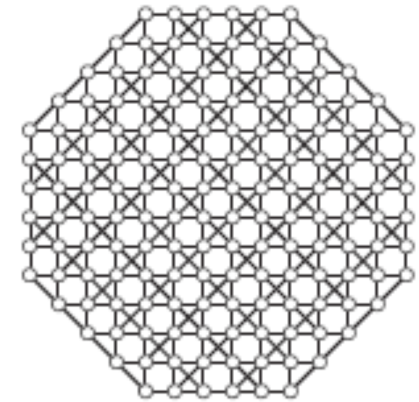


$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i S_i^z$$

(b)

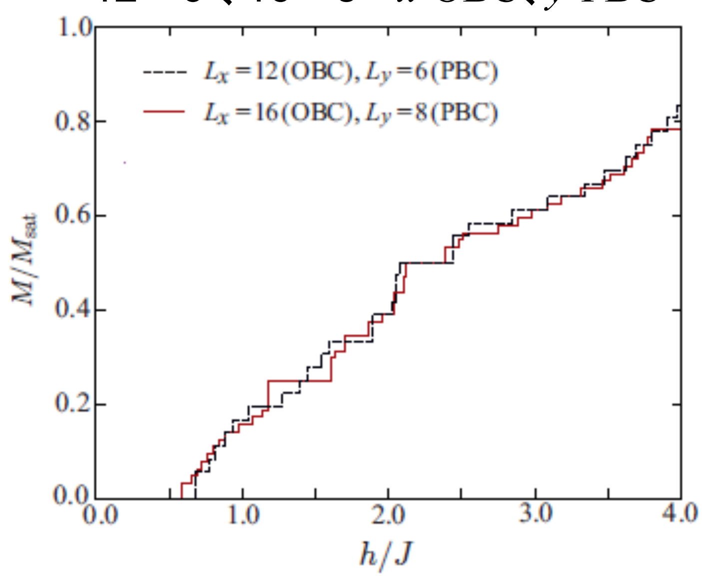


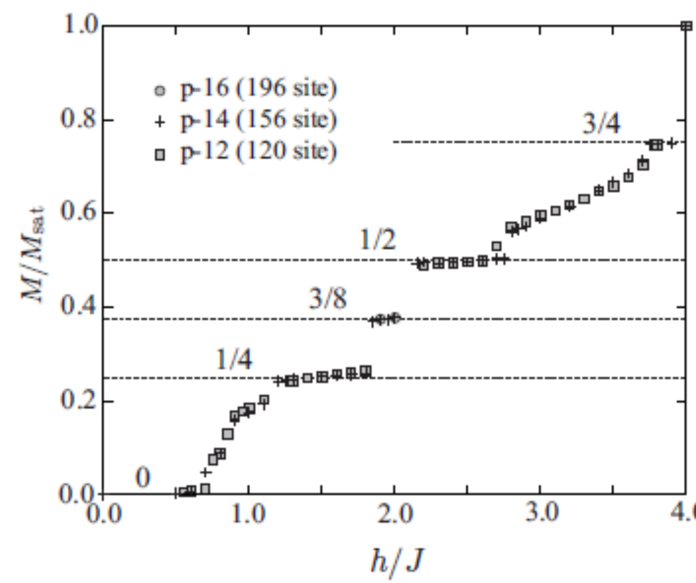
p-14



Morita and Shibata: PRB 94 140404 (2016)

12×6 、16×8　x-OBC、y-PBC

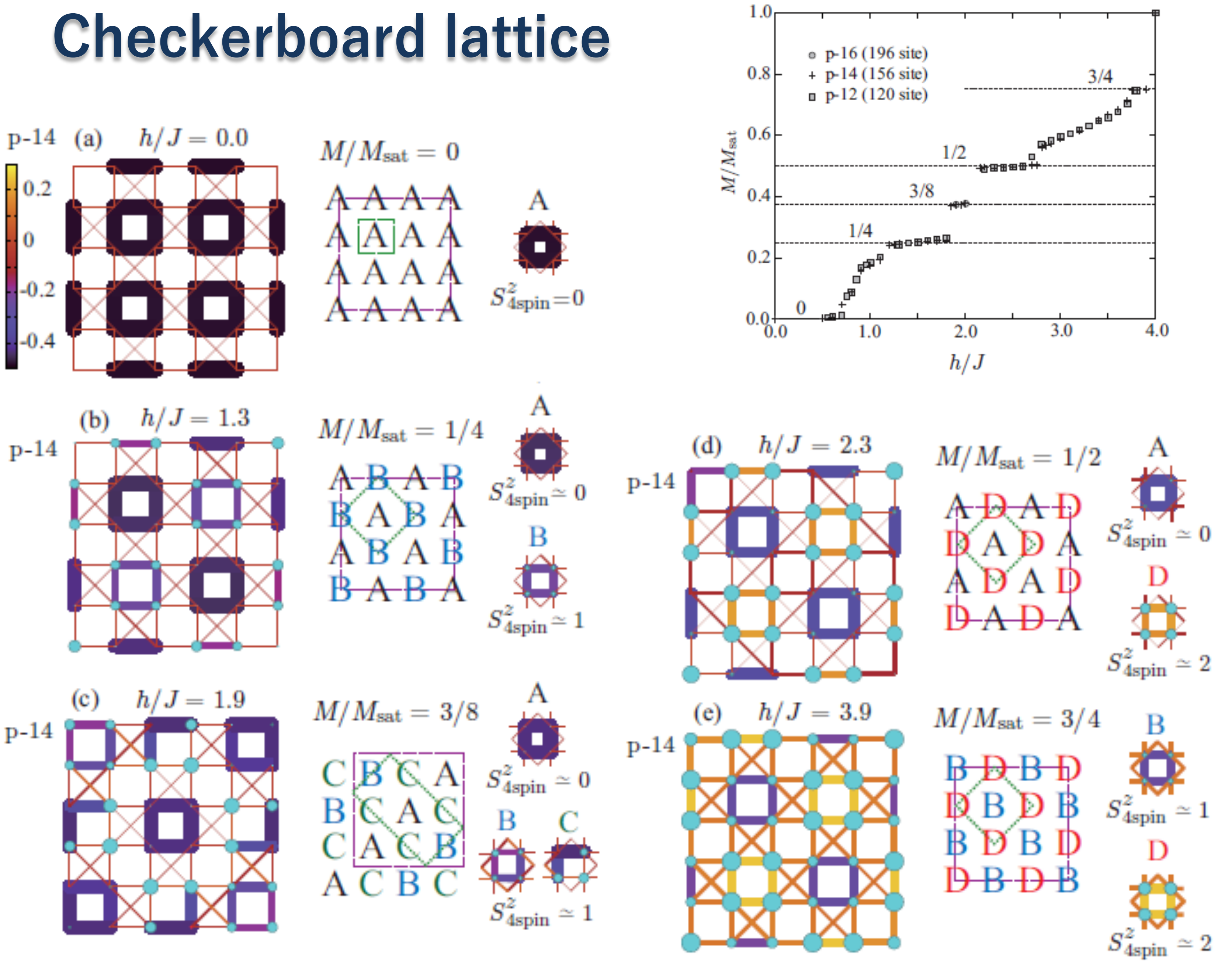




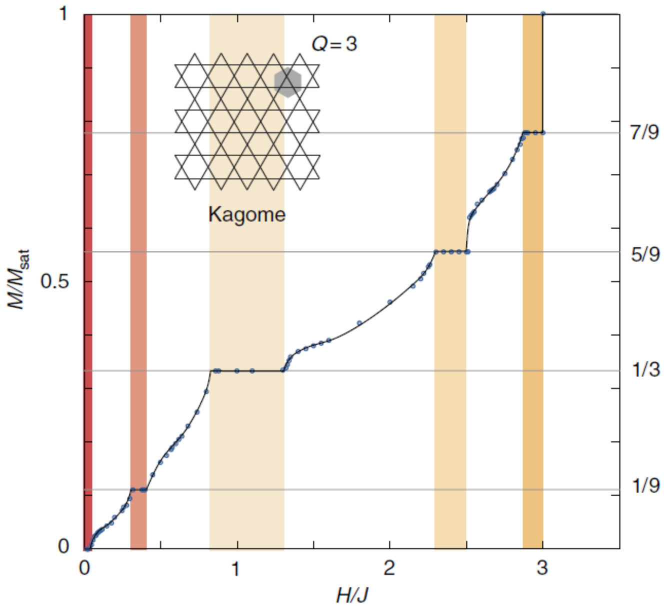
new

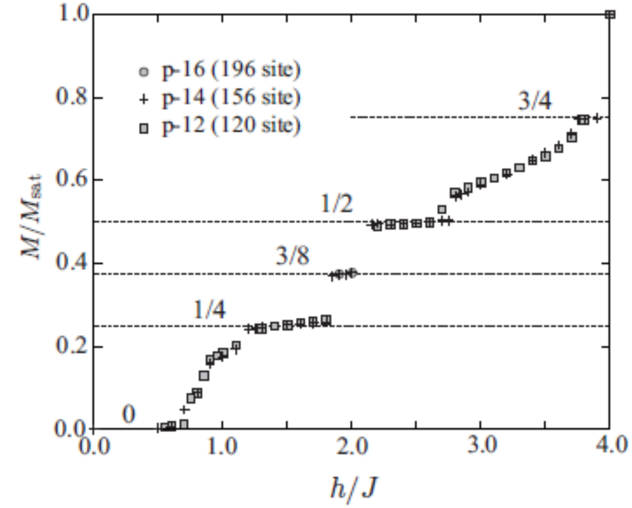
new

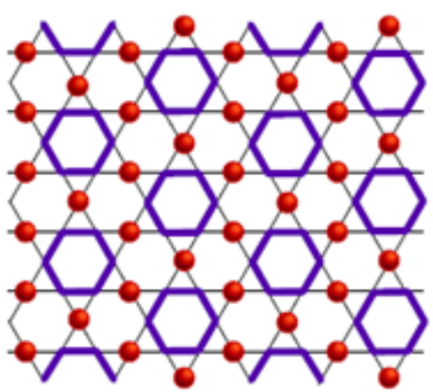
Checkerboard lattice

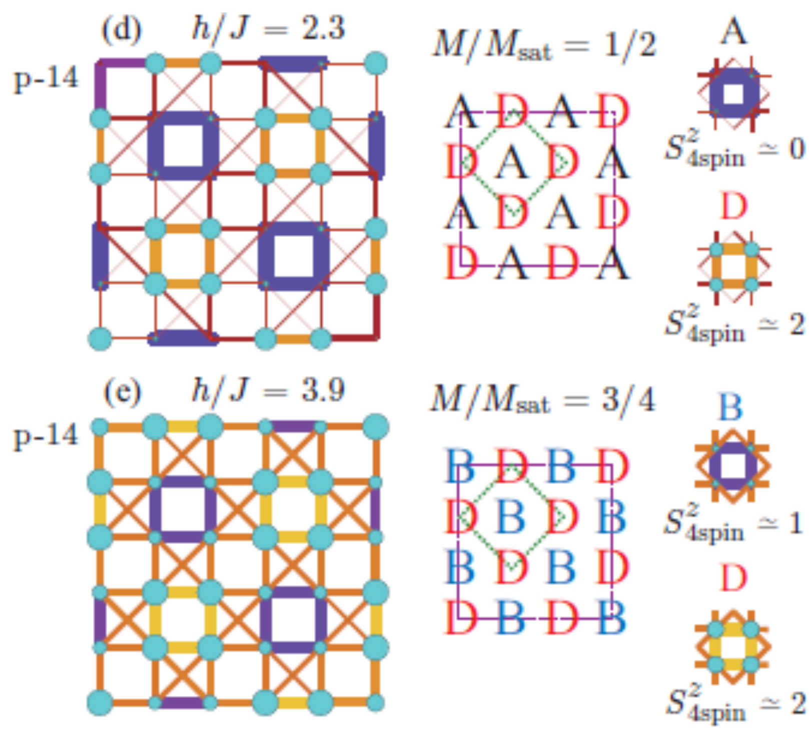


Magnetization plateau





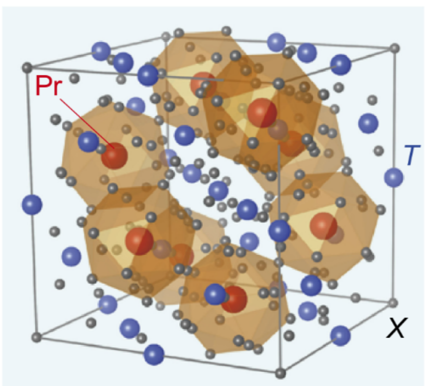


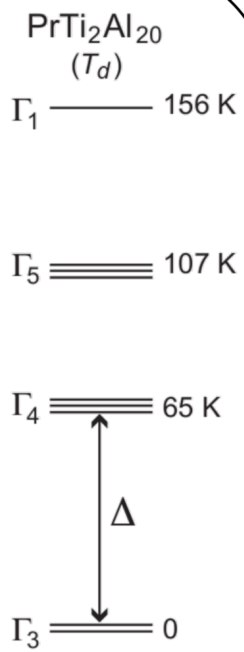


Two-channel Kondo lattice model

Pr籠型化合物 ※

$\mathrm{Pr}\,T_2X_{20}$





Prのf^2配置が非Kramers二重項

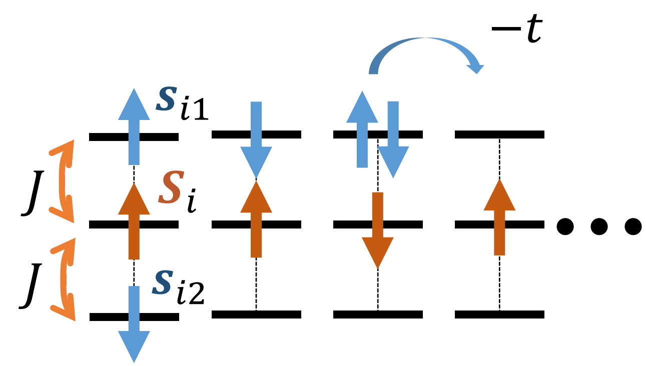
f-電子の軌道自由度と伝導電子の混成模型

$$H=-t\sum_{i\sigma\alpha}\left(c^{\dagger}_{i\alpha\sigma}c_{i+1,\alpha\sigma}+\mathrm{h.c.}\right)+J\sum_{i\alpha}\boldsymbol{s}_{i\alpha}\cdot\boldsymbol{S}_i$$

伝導電子($\alpha=1$)

局在スピン

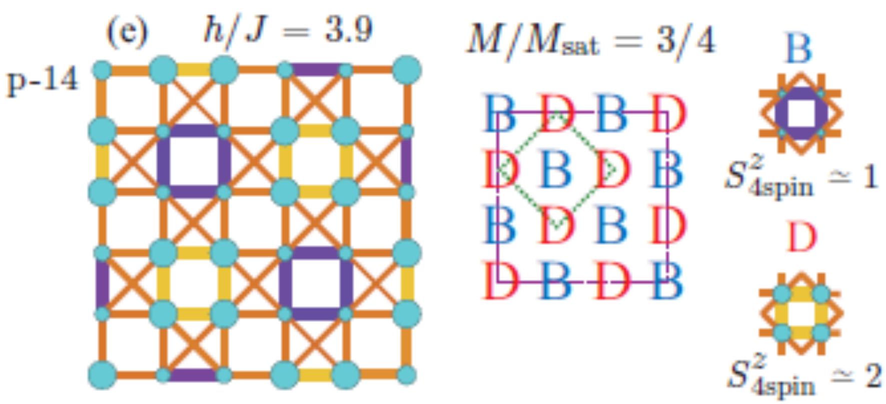
伝導電子($\alpha=2$)



※ T. Onimaru and H. Kusunose, J. Phys. Soc. Jpn. **85**, 082002 (2016)

まとめ

- SSD により、境界条件の影響を効果的に抑制できる
  （並進対称性、外場応答、サイズ外挿）
- 局所的量子もつれが生み出す安定秩序相
  （磁化プラトー相、ダイマー絶縁相）
- 複数の自由度の協調による量子相の形成
  （軌道自由度がスピンギャップ相、絶縁相への転移を促進）

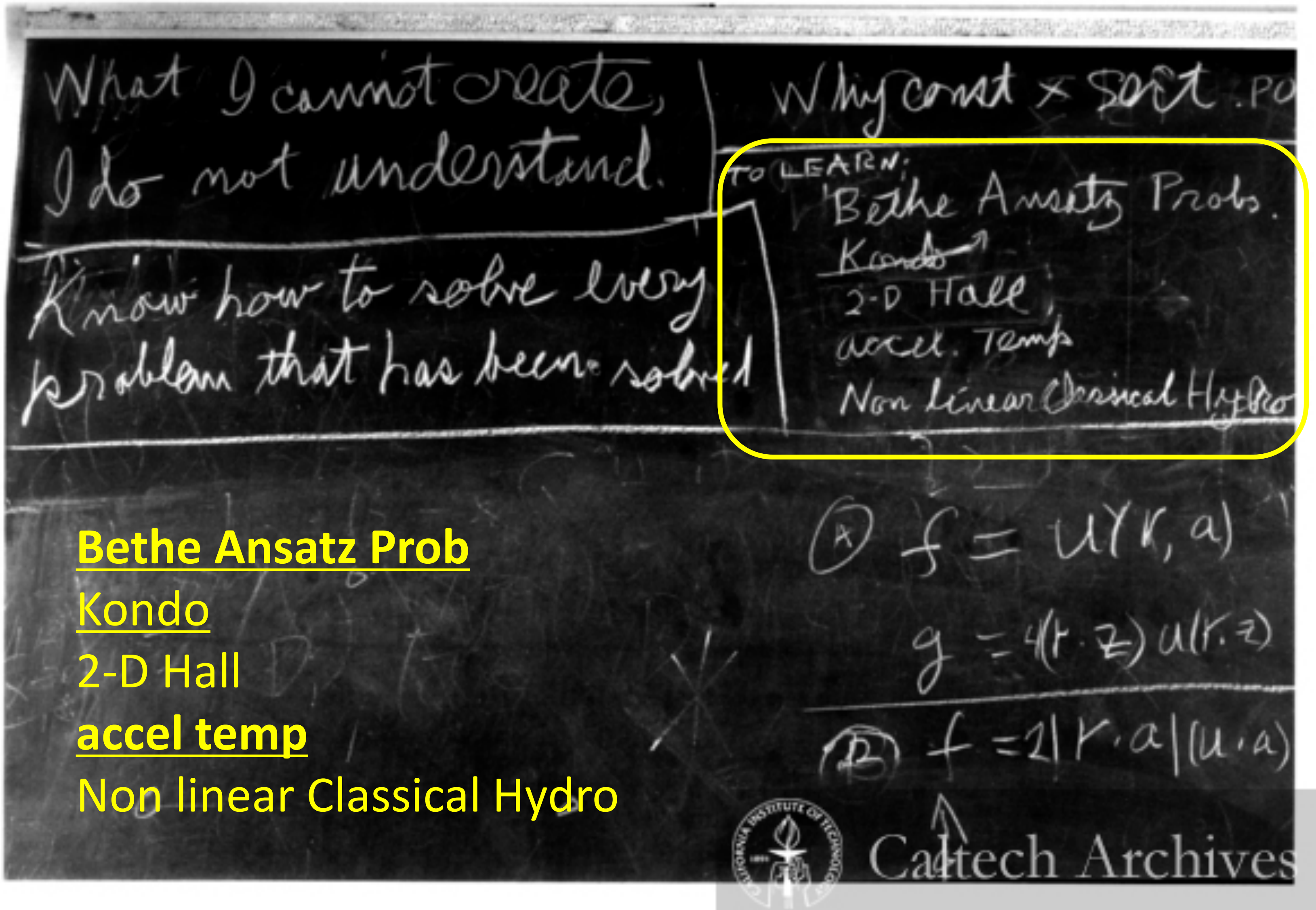


SSD@理研:2019/7/11

XXZ鎖における格子Unruh効果と世界線エンタングルメント

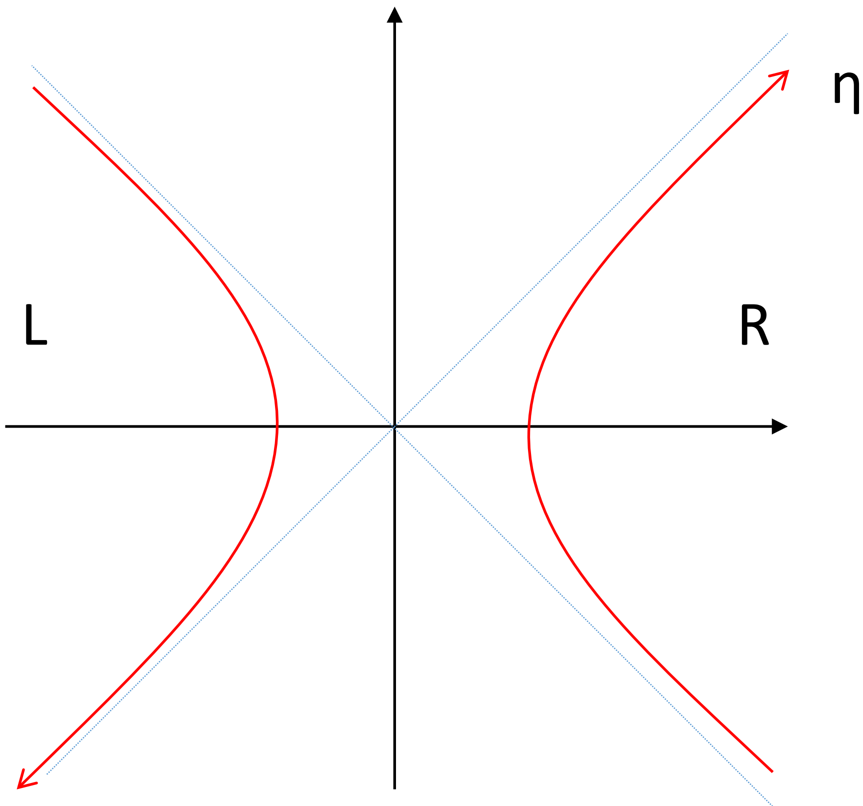
新潟大理　奥西巧一, 関孝一

Feynman's blackboard at 1988



Bethe Ansatz Prob
Kondo
2-D Hall
accel temp
Non linear Classical Hydro

Unruh effect



A constantly accelerating observer

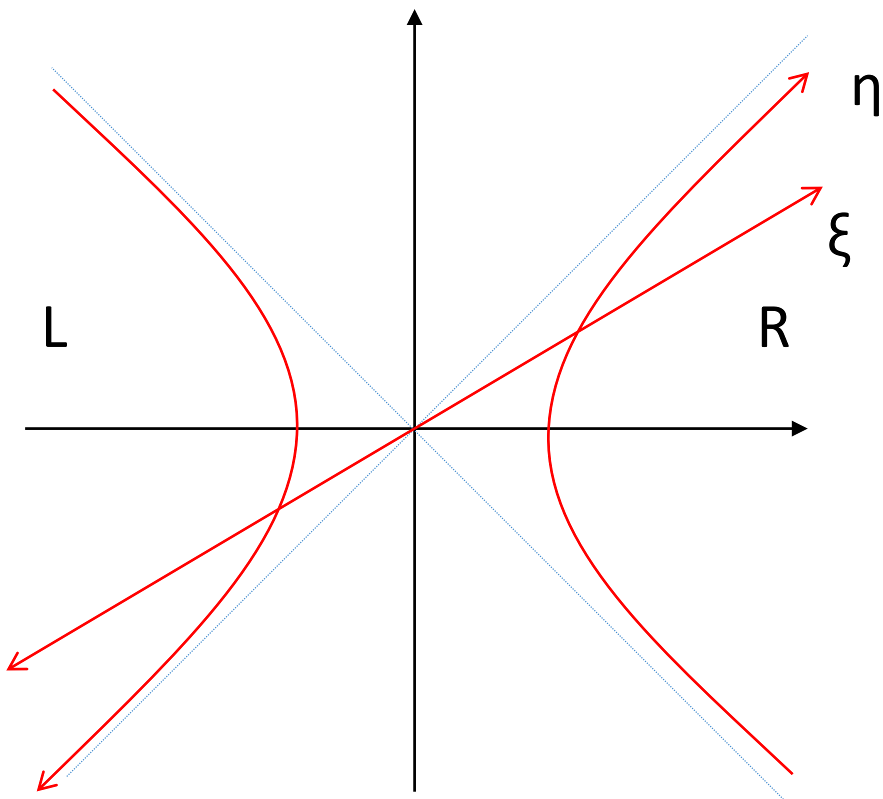
$$
\begin{aligned}
x &= \frac{e^{a\xi}}{a}\cosh(a\eta) \\
t &= \frac{e^{a\xi}}{a}\sinh(a\eta)
\end{aligned}
$$

sees the vacuum as a thermalized state with an effective temp. (Unruh temp.)

$$\beta^* = \frac{2\pi}{a}$$

The Left and right parts are space like regimes, which are classically separable!

Rindler-Fulling quantization



constantly accelerating observer

R $$x = \frac{e^{a\xi}}{a}\cosh(a\eta)$$
$$t = \frac{e^{a\xi}}{a}\sinh(a\eta)$$

$a_k \, , a_k^\dagger$ with $a_k|0\rangle_M$

Minkowski vacuum

$b_p^{\mathrm{R}} \, , {b^{\mathrm{R}}}_p^\dagger \, , b_p^{\mathrm{L}} \, , {b^{\mathrm{L}}}_p^\dagger$ with

$$b_p^{\mathrm{R}}|0\rangle_R = b_p^{\mathrm{L}}|0\rangle_L = 0$$

Bogoliubov transformation

$$|0\rangle_M = e^{-\prod_p e^{-\pi p/a} {b^L}_p^\dagger {b^R}_p^\dagger}|0\rangle_L|0\rangle_R$$

$$\rho_R = \mathrm{Tr}_L|0\rangle_M\langle 0|_M = \prod_p e^{-\beta^* H_p}$$

with $\beta^* = \dfrac{2\pi}{a}$ and $H_p = p{b^R}_p^\dagger b_p^R$

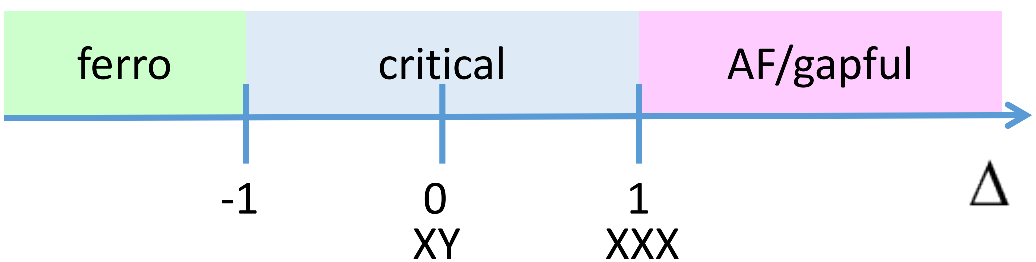
Ising-like XXZ chain

$$\lambda > 0 \quad (\Delta > 1)$$

$$\mathcal{H} = J_\lambda \sum_{n=-L+1}^{L} \left[S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z \right]$$

$$J_\lambda = \frac{2}{\sinh \lambda} \qquad \Delta = \cosh \lambda$$

The groundstate is gapful with a finite correlation length.

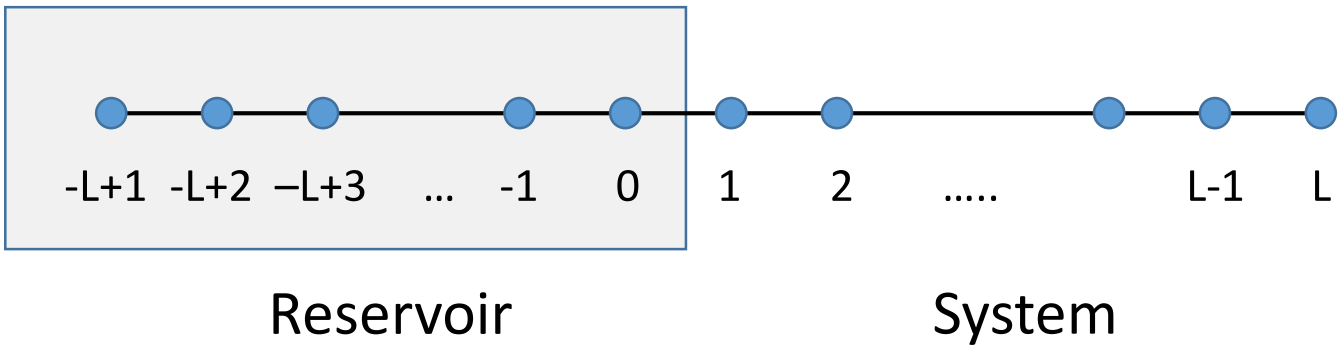


<u>Bethe ansatz solvable</u>

Bulk energy, excitation gap, magnetization, etc.

But, direct computation of the Bethe wavefunction is not so useful

geometry for EE



$$\rho = \sum_{n_-} \Psi^\dagger(n_-, n_+)\Psi(n_-, n_+) \quad \Rightarrow \quad S = -\mathrm{Tr}[\rho \log \rho]$$

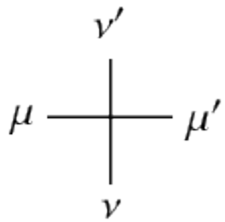
* This bipartition EE can be easily calculated by DMRG.

If we can write $\rho \sim \exp\left(-H_{\mathrm{EE}}\right)$, H_{EE} is called entanglement Hamiltonian or modular Hamiltonian.

A modular Hamiltonian defines a time evolution in an angular direction different from the conventional time.

XXZ chain and 6-vertex model

$$W(\mu,\nu|\mu',\nu') =$$

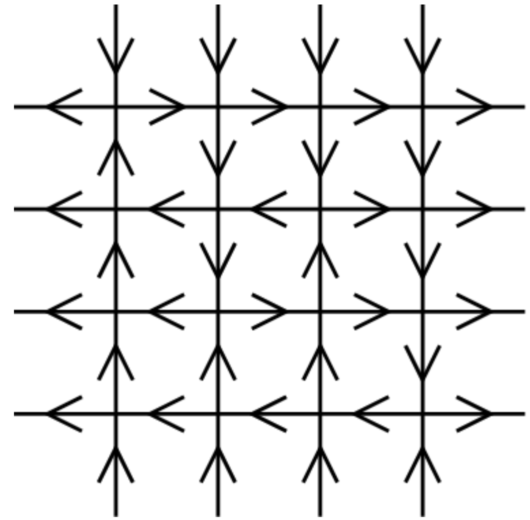


$$W(+,+|+,+) = W(-,-|-,-) = 1$$

$$W(+,-|-,+) = W(-,+|+,-) = \frac{\sinh(u)}{\sinh(\lambda - u)}$$

$$W(+,-|+,-) = W(-,+|-,+) = \frac{\sinh(\lambda)}{\sinh(\lambda - u)}$$

satisfies Yang-Baxter relation



$\lambda > 1$ Ising-like anisotropy = antiferroelectric regime

Commuting transfer matrices

$$[T(u), T(u')] = 0 \qquad T(u) = \sum_{\{\mu\}} \prod_n W_n(\mu_n, \nu_n | \mu_{n+1}, \nu_{n+1})$$

u : rapidity(=spectral parameter)

Hamiltonian of the XXZ chain $\quad \mathcal{H} = -\frac{d}{du} \log T(u)\Big|_{u=0}$

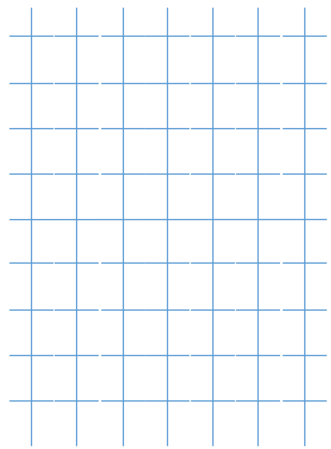
Simultaneous eigenstate $\quad [T(u), \mathcal{H}] = 0$

integrability and CTM

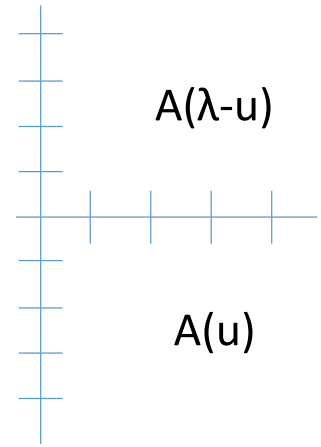
Eigenvector: ~~Bethe type~~

Baxter's magic / CTM

$$|\Psi\rangle \sim \lim_{n\to\infty} T^n |b\rangle$$ =



=



Baxter, J.Math.Phys. (1968), J.Stat.Phys. (1971)

The groundstate wavefunction of H can be written as a product of CTMs

$$\Psi \sim A(\lambda - u)A(u) \quad \text{with} \quad A(u) \sim e^{-u\mathcal{K}}$$

$\mathcal{K}$ Hamiltonian of corner transfer matrix/corner Hamiltonian

$$\mathcal{K} \equiv J_\lambda \sum_{n=1}^{L} n \left\{ S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z \right\}$$

Lattice Lorentz boost operator
(Rapidity shift operator)

$$A(-\mu) T(\nu) A(\mu) = T(\mu + \nu)$$

$\Rightarrow$ The CTM formulation corresponds to the Rindler quantization of the relativistic quantum field theory

Lattice Poincare algebra

H.B.Thacker, Physica D **18**, 348 (1986).

$$[P, \mathcal{H}] = 0\,, \quad [\mathcal{K}, P] = iH\,, \quad [\mathcal{K}, H] = i\tilde{I}_2$$

$$I_0 = iP \quad I_1 = -\mathcal{H} \qquad \tilde{I}_2 = iI_2 = \sum [h_{n,n+1}, h_{n+1,n+2}] \qquad \log T(u) = \sum \frac{I_n}{n!} u^n,$$

<u>Reduced density matrix</u>

$\mathcal{K}$ plays a role of the entanglement Hamiltonian

$\Rightarrow$ $\rho = \exp(-\beta_\lambda \mathcal{K})/Z$ with $\beta_\lambda \equiv 2\lambda$, $Z \equiv \mathrm{Tr} \exp(-\beta_\lambda \mathcal{K})$

entanglement/corner Hamiltonian

$$\mathcal{K} \equiv J_\lambda \sum_{n=1}^{L} n \left\{ S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + \Delta S^z_n S^z_{n+1} \right\}$$

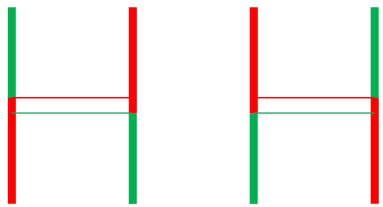
Free boundary condition at n=1, L

The boundary effect at n=1 should be perfectly suppressed

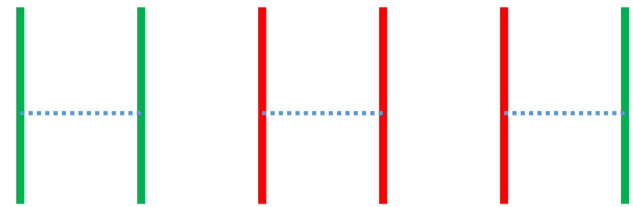
The energy scale is proportional to n

Effective temperature decreases as n increases.
(This can be a source of difficulty in a QMC simulation)

WL QMC



off-diagonal interaction
(XY-terms)



diagonal interaction
(zz terms)

The energy scale is proportional to n. ⇨ We draw WLs as circles
"classical" entanglement surrounding the entangle point

Scale imaginary time: τ $\quad \theta = a\tau \quad$ with $\quad a = \dfrac{2\pi}{\beta_\lambda} = \dfrac{\pi}{\lambda}$

$0 \leq \tau < \beta_\lambda$

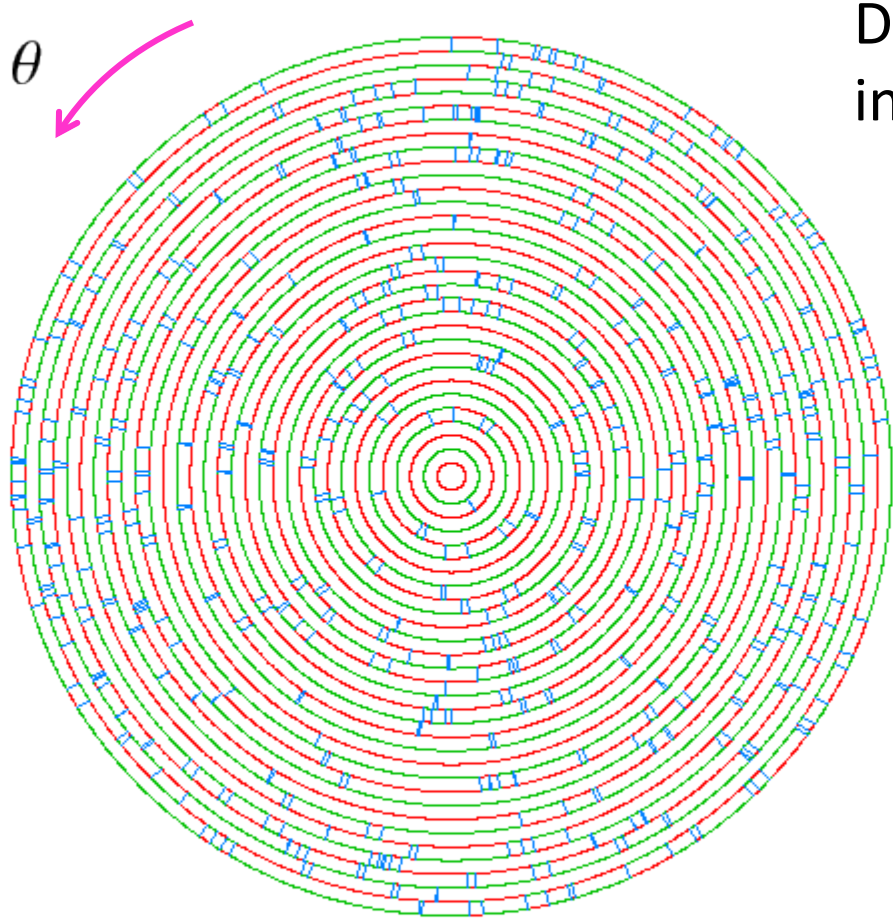
a : effective acceleration

a=0 : classical limit

snapshots

$$\Delta = 2.0 \qquad \beta_\lambda \equiv 2\lambda$$
$$(\lambda = 1.3169\cdots)$$

How can the “uniform” ground state be realized for the non uniform Hamiltonian?



Density of kinks looks uniform in this plot!

Local temp $\propto n\beta_\lambda$

$\Updownarrow$

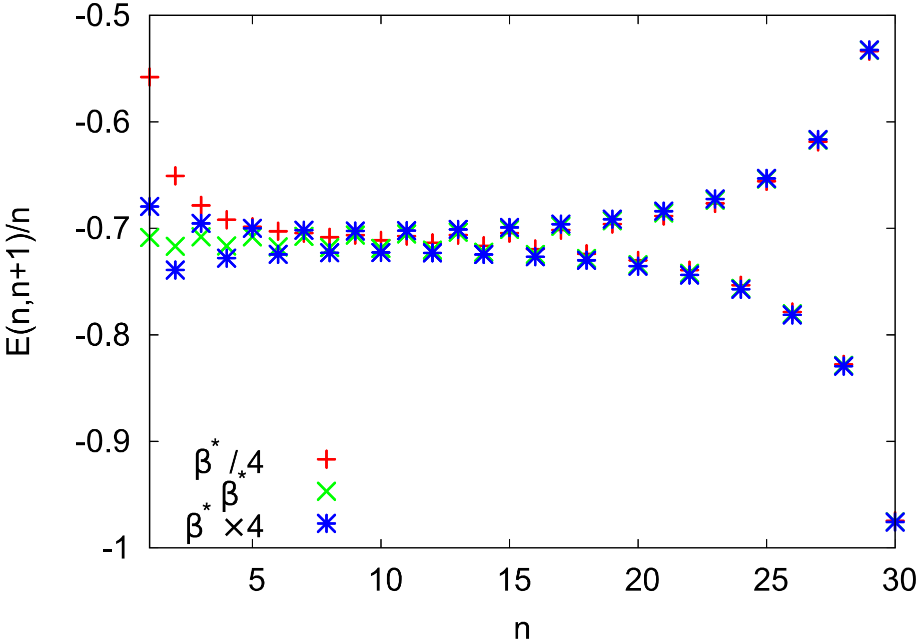
kink # $\propto n\beta_\lambda$

For $\beta < \beta_\lambda$ (high temp.), kinks around the center becomes space.

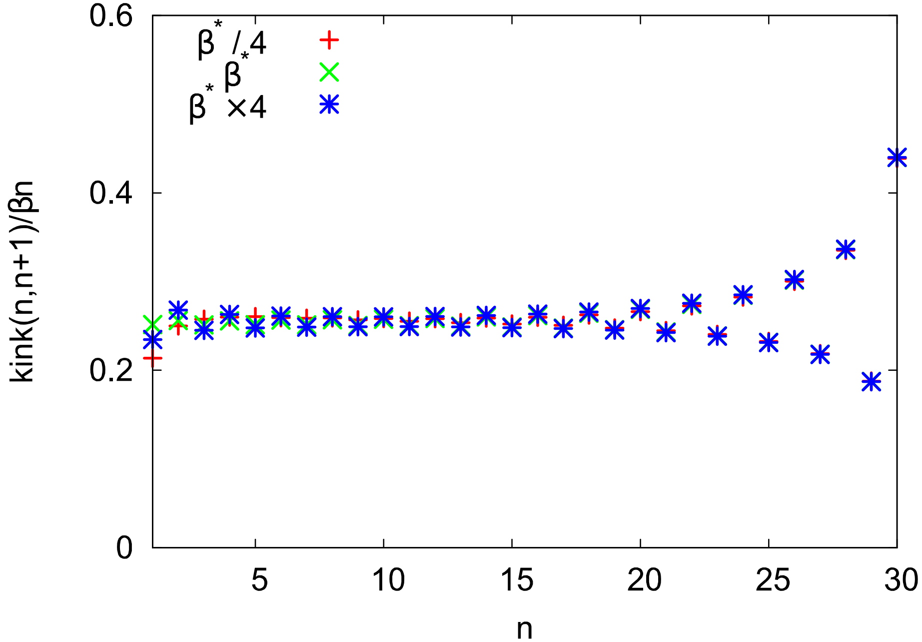
For $\beta > \beta_\lambda$(low temp.), kinks around the center are oscillating.

bond energy distribution $\Delta = 2.0$

$\langle S_n \cdot S_{n+1} \rangle$ normalized bond energy



normalized kink density

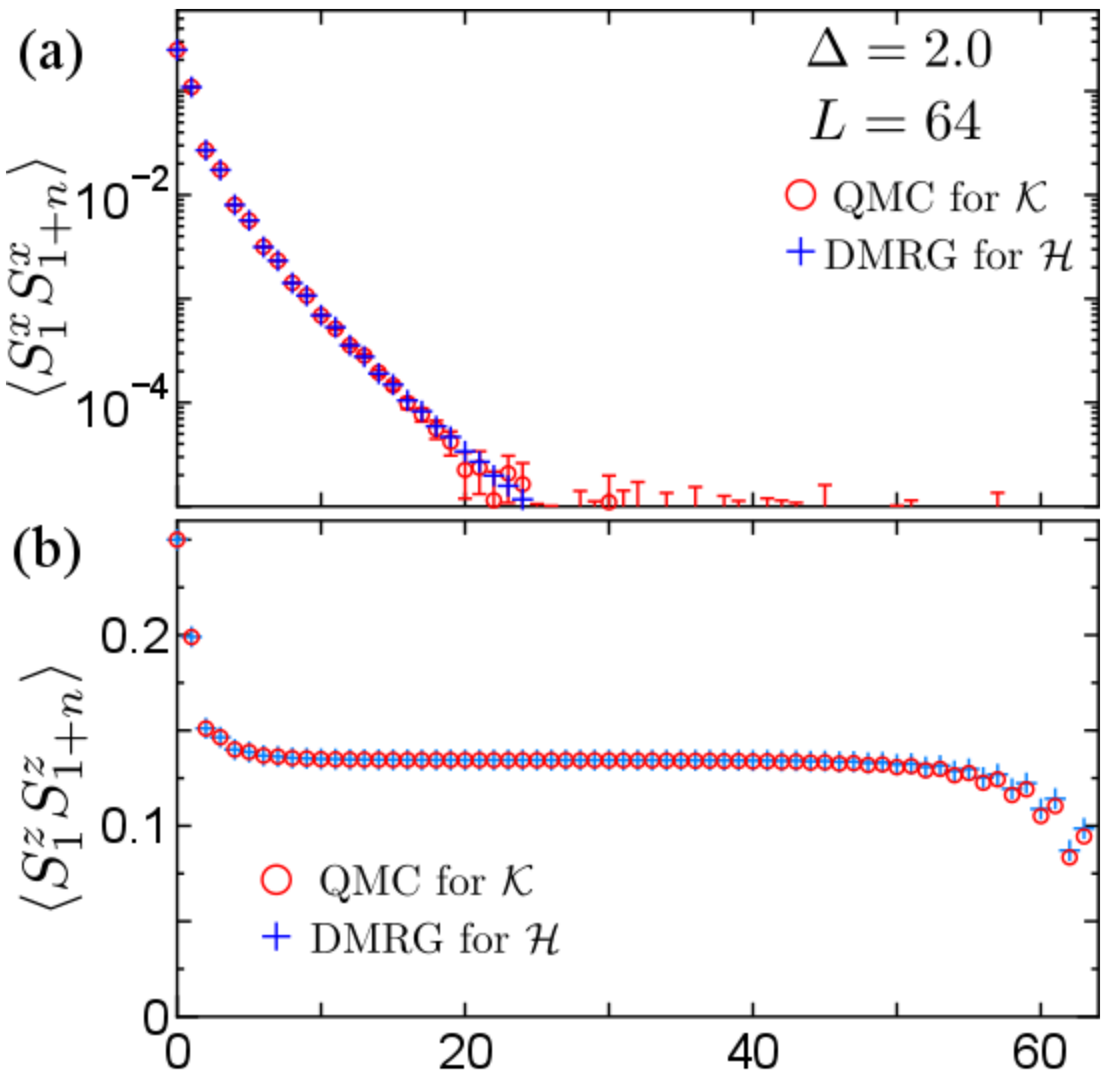


kink density is related to the off-diagonal parts of local energy

At $\beta = \beta^*$, the normalized bond energy and kink density become flat around n=1

reproduces uniform ground state wavefunction.

correlation functions $\Delta = 2.0$



Perfect correspondence to the DMRG results for the groundstate of H

Entanglement Entropy

The groundstate entanglement entropy for H can be calculated as the thermal entropy for the entanglement Hamiltonian.

$$S_{\mathrm{EE}} = -\mathrm{Tr}_S[\rho \log \rho] = \beta_\lambda \langle \mathcal{K} \rangle + \log Z$$

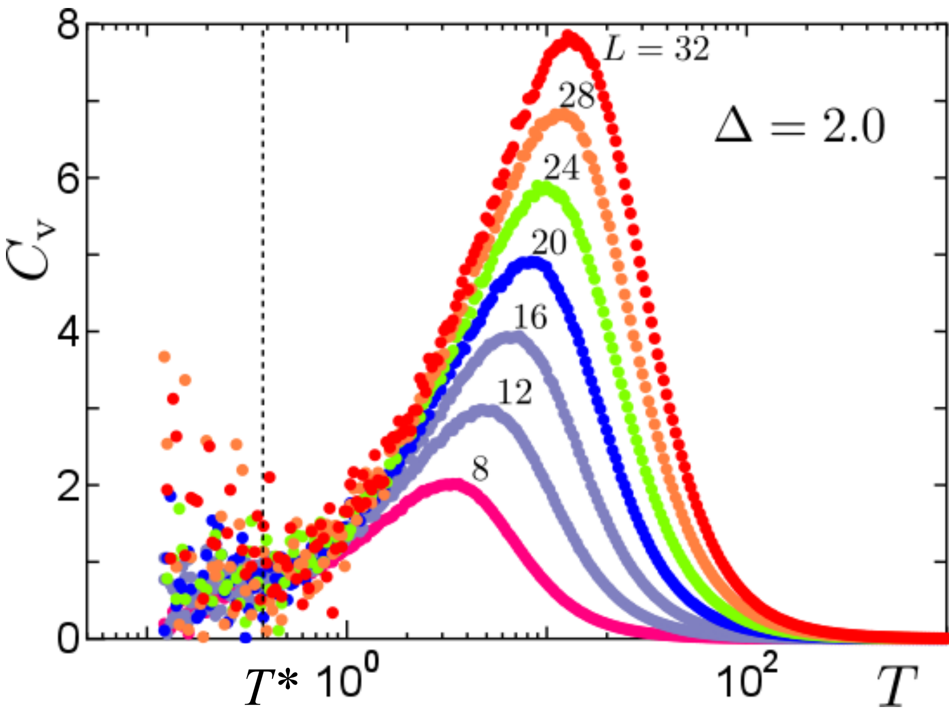
We calculate S_EE with integration of a specific heat estimated by a QMC simulation.

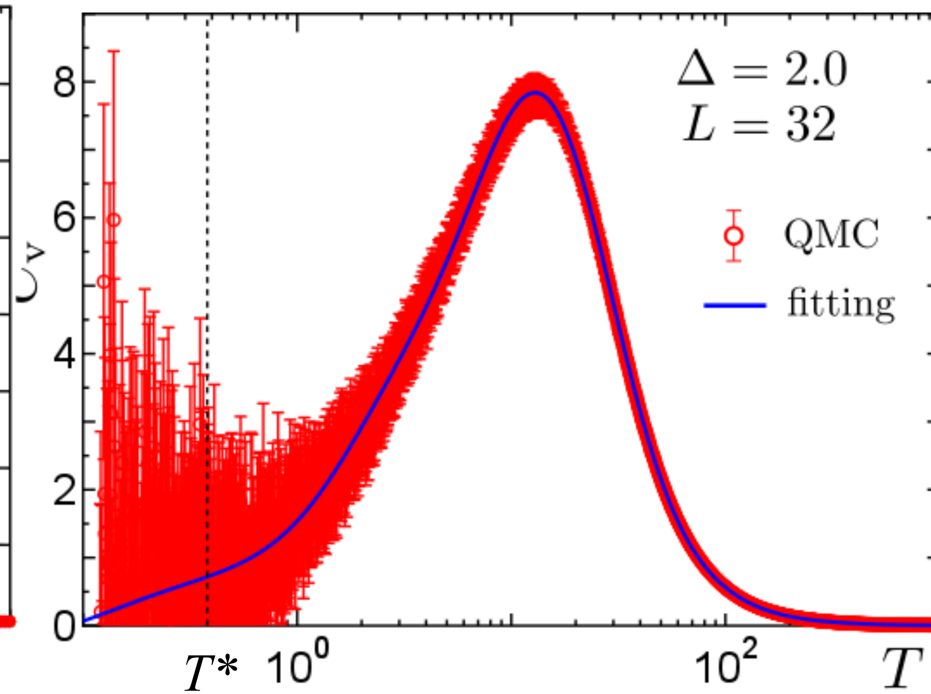
$$S_{\mathrm{EE}} = L \log 2 - \int_{T_\lambda}^{\infty} \frac{C_{\mathrm{v}}}{T} dT = L \log 2 - \int_{\log T_\lambda}^{\infty} C_{\mathrm{v}} dx$$

The estimation of the entropy is not easy but possible with QMC.

Cv

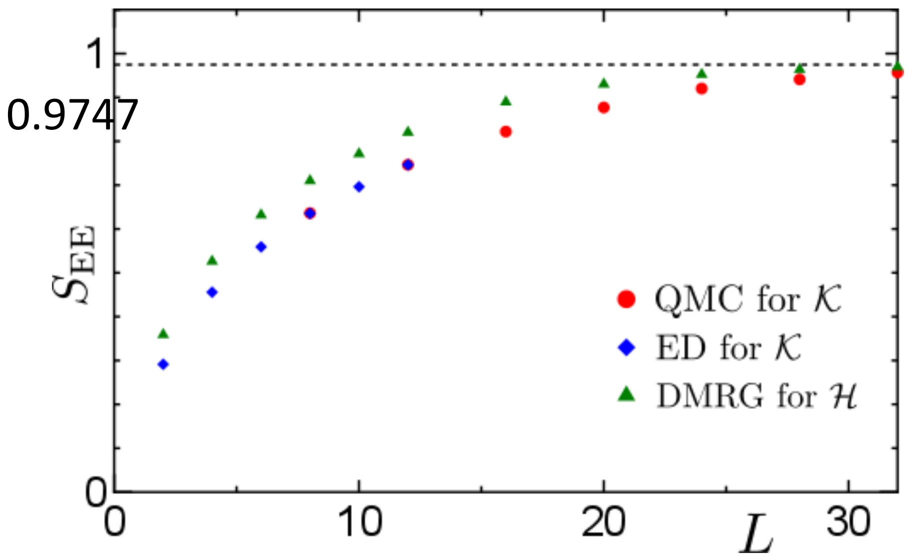
Fitting: Gaussian Kernel method



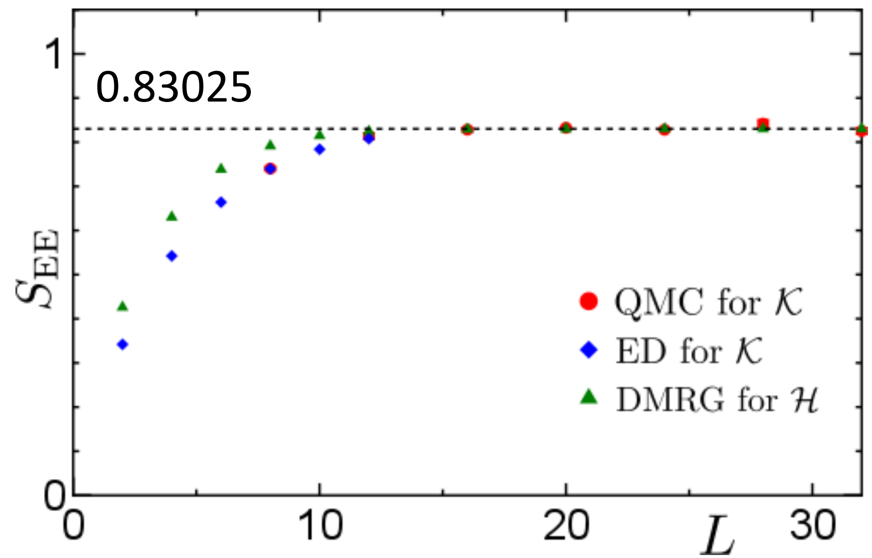


Eneanglement Entropy

$\Delta = 2.0$



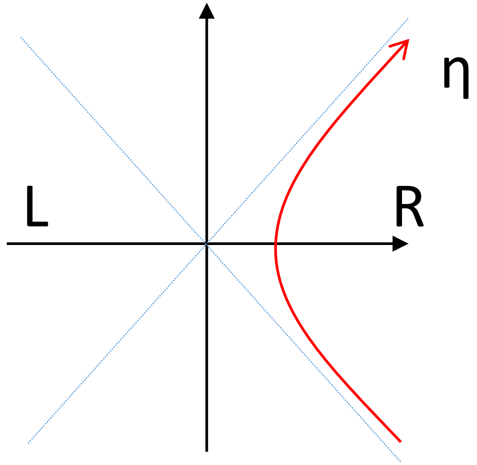
$\Delta = 3.0$



Estimation of EE approaches to the exact value of EE for the half0infinite subsystem

The deviation from the DMRG result originates from geometry of world sheets:
DMRG: cylinder, corner Hamiltonian: annuls

Unruh-DeWitt detector



A harmonic oscillator coupled with a scalar field moving along the Rindler trajectory

$$S = \int d\eta \phi(x(\eta), t(\eta)) \hat{X}(\eta)$$

$$x = r\cosh(a\eta)\,,\, t = r\sinh(a\eta)$$

➡ The detector is excited by the thermalized vacuum.

Excitation rate is given by an integration of the Wightman function

➡ $$P_n \propto \int d\eta e^{i\omega_n \eta} {}_M\langle \phi(x(\eta), t(\eta))\phi(r, 0)\rangle_M$$

Capturing the Bose distribution with the Unruh temp.

$$P_n \propto \frac{1}{e^{\beta_U \omega_n} - 1}$$

(massless case)

XXZ-chain analogue of the detector

A harmonic oscillator coupled with a spin in the XXZ chain

But, the detector does not accelerate in the chain literally .

<u>Scalar field</u> $$\phi(x(\eta), t(\eta)) = e^{ia\eta L}\phi(r, 0)e^{-ia\eta L}$$

η-dependent Lorentz transformation

Spin coupled with the detector : $\mathcal{K}$ lattice Lorentz boost

$$\Rightarrow \quad S_n^{\mu}(\eta) = e^{-ia\eta\mathcal{K}} S_n^{\mu} e^{ia\eta\mathcal{K}}$$

$n \sim r$: distance from the entangle point

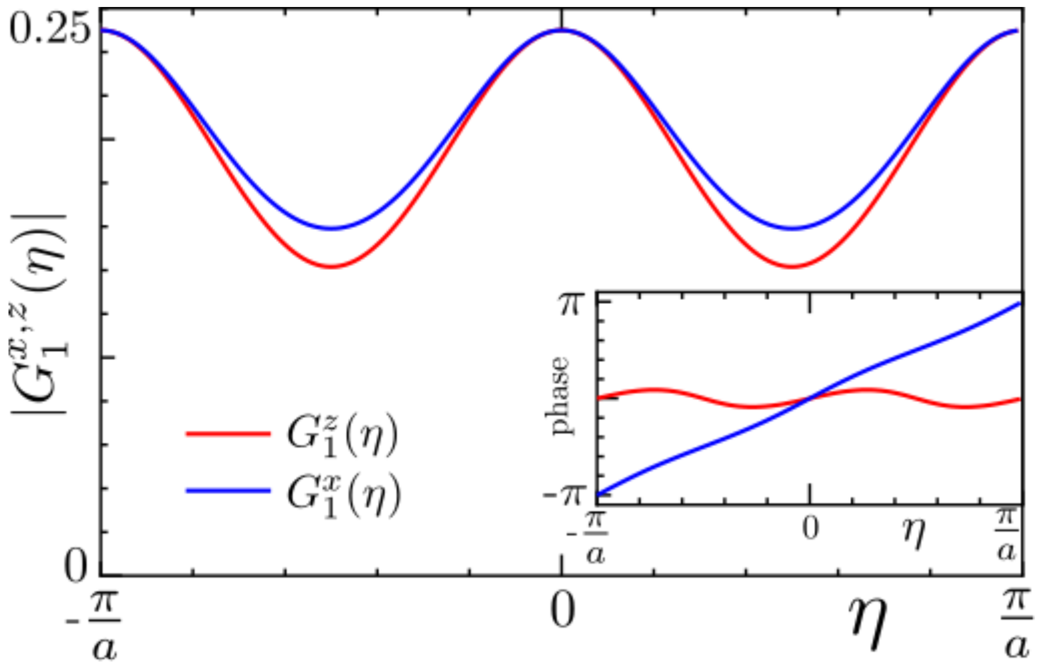
Autocorrelation function with respect to τ

$$G_n^{\mu}(\eta) \equiv \frac{\mathrm{Tr}\, S_n^{\mu}(\eta) S_n^{\mu}(0) e^{-\beta_\lambda \mathcal{K}}}{Z}$$

Autocorrelations

DMRG: Renormalization transformation matrix gives
the relation between the $\mathcal{K}$ diagonal bases and the usual spin bases
Bogoliubov trans. (Rindler) (Minkowski)

n=1



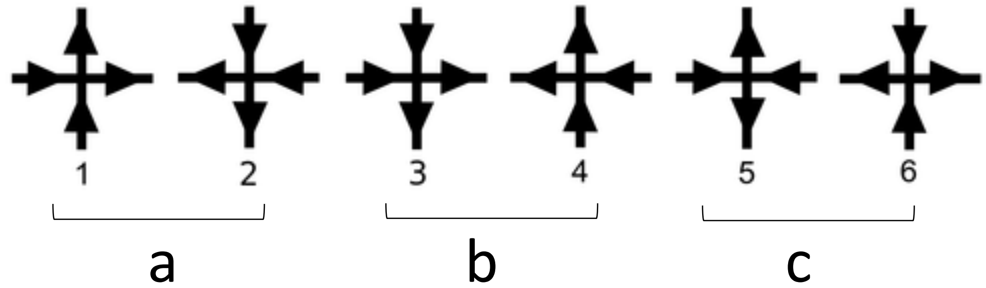
classical value

π/a periodicity

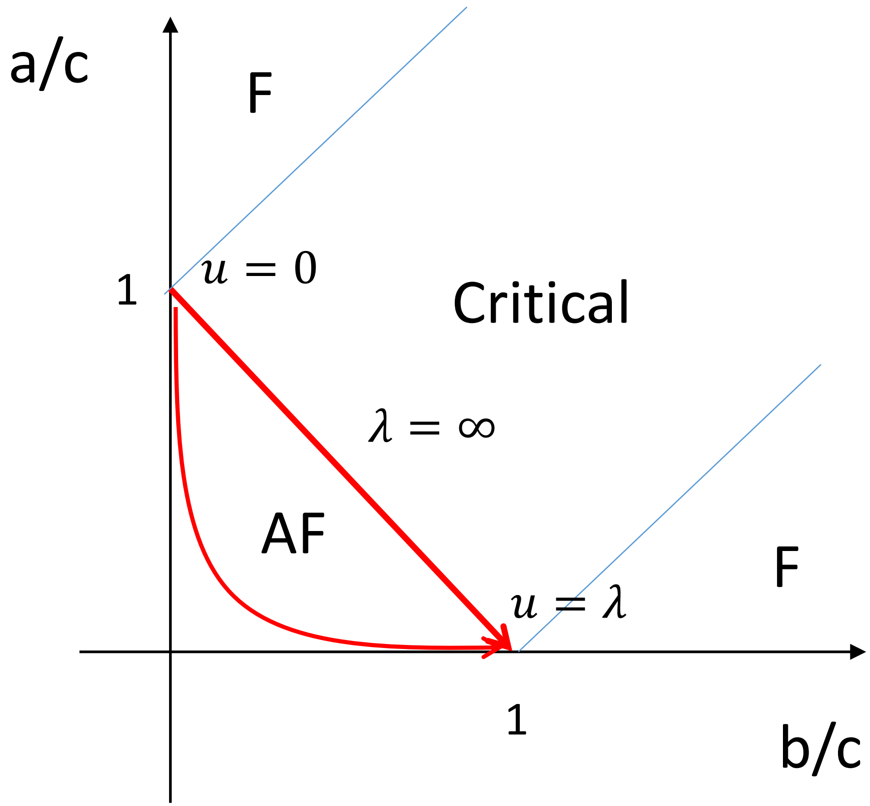
Imaginary shift
of the rapidity
+
lattice effect

rapidity space and 6-vertex model

(spectral parameter)



Phase diagram



$$0 < \Re u < \lambda \to 2\lambda$$

$$0 < \tau < \frac{2\pi}{a}$$

Imaginary part: Bethe ansatz

$$e^{ik} = \frac{\sinh((\lambda + i\alpha)/2)}{\sinh((\lambda - i\alpha)/2)}$$

$$\alpha = a\eta \quad \Rightarrow \quad -\frac{\pi}{a} < \eta < \frac{\pi}{a}$$

summary/discussions arXiv:1906.10441

- We calculate the groundstate properties of the Ising-like XXZ chain with a finite temperature formulation based on the entanglement Hamiltonian/CTM.

Lattice Unruh effect

- We can understand the entanglement from the viewpoint of classical world lines surrounding the entangle point

world-line entanglement

- Can we realize lattice Unruh-Dewitt detector?

entanglement detector

- Critical cases? CFT, SSD, numerically bad convergence

Perspectives from Sine-square deformation on conformal field theories

Tsukasa TADA

arXiv:1504.00138
1602.01190
1712.09823
1904.12414

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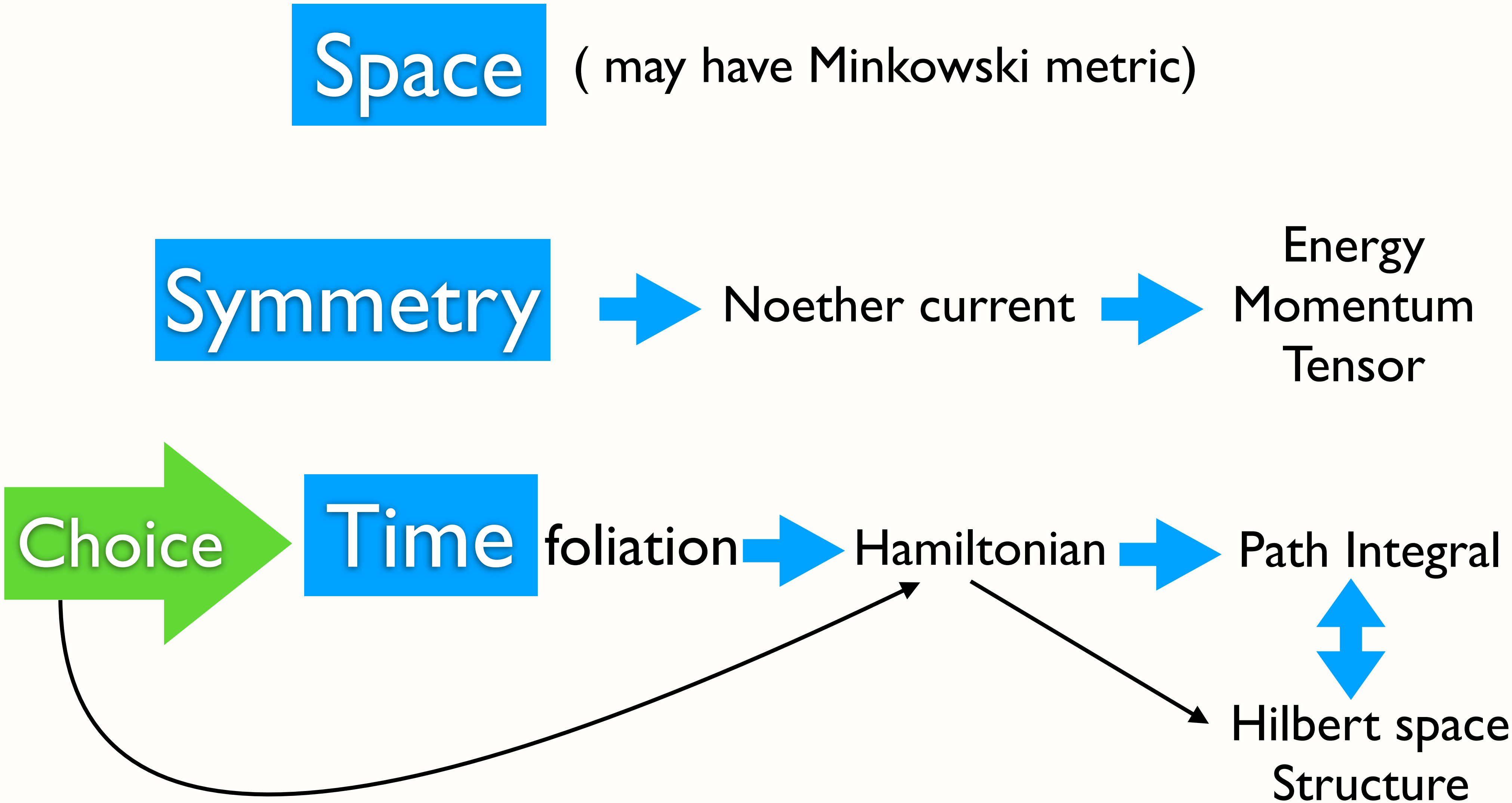

Space
(may have Minkowski metric)
Symmetry
Noether current
Energy Momentum Tensor
Choice
Time
foliation
Hamiltonian
Path Integral
Hilbert space Structure

Principle for

Choice → **Time** foliation

↓ respect

Symmetry → Conservation of "Energy" (guaranteed)

↓

Unitarity

Hilbert space Structure ↔ Unitarity

What

Symmetry

Conformal symmetry

$$\bigcup$$

$$SL(2,\mathbb{R})\left(\times SL(2,\mathbb{R})\right)$$

Choice

Classified by

Casimir invariant

What

Symmetry Conformal symmetry

Space accomodates $SL(2,\mathbb{R})\left(\times SL(2,\mathbb{R})\right)$

$$\mathbb{S}^2 \qquad \mathbb{H}^2$$

$\mathbb{AS}_3$ → Ryu-san's talk

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$

2) $\mathcal{H} = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2}$

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

Casimir

$$c^{(2)} \equiv \left(x^{(0)}\right)^2 - 4x^{(1)}x^{(-1)} \qquad \Longleftrightarrow \qquad x^{(0)}L_0 + x^{(1)}L_1 + x^{(-1)}L_{-1}$$

adjoint action

$$\left(x'^{(0)}\right)^2 - 4x'^{(1)}x'^{(-1)} \qquad\qquad x'^{(0)}L_0 + x'^{(1)}L_1 + x'^{(-1)}L_{-1}$$

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$

$$\boxed{c^{(2)} > 0}$$

2) $\mathcal{H} = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2}$

$$\boxed{c^{(2)} = 0}$$

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$$\boxed{c^{(2)} < 0}$$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$ radial quantization

2) $$\mathcal{H} = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2}$$

dipolar quantization Continuous Virasoro algebra

Sine-square Deformation

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

$$\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$$

$$H[f] = \int dx\, f(x)\mathcal{H}(x).$$

Envelope function

TABLE I. Summary of conformal maps and deformed evolution operators discussed in the main te

	Conformal map	"Time"	"Space"	Envelope function
Angular quantization	$w = \ln z$	v	u	$f(x) = x$
Radial quantization	$w = \ln z$	u	v	$f(s) = \frac{1}{L}$
Entanglement Hamiltonian	$w = \ln \frac{(z+R)}{(z-R)}$	v	u	$f(x) = \frac{(x-R)(x+R)}{2R}$
Regularized SSD (rSSD)	$w = \ln \frac{(z+R)}{(z-R)}$	u	v	$f(s) = \cos \frac{2\pi s}{L} + \cosh u_0$
Sine-square deformation (SSD)	$w = \frac{1}{z}$	u	v	$f(s) = \sin^2 \frac{\pi s}{L}$
Square root deformation (SRD)	$z = \sin w$	v	u	$f(x) = \sqrt{x^2 - R^2}$

Entanglement

TT, arXiv: 1904.12414

$$\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$$

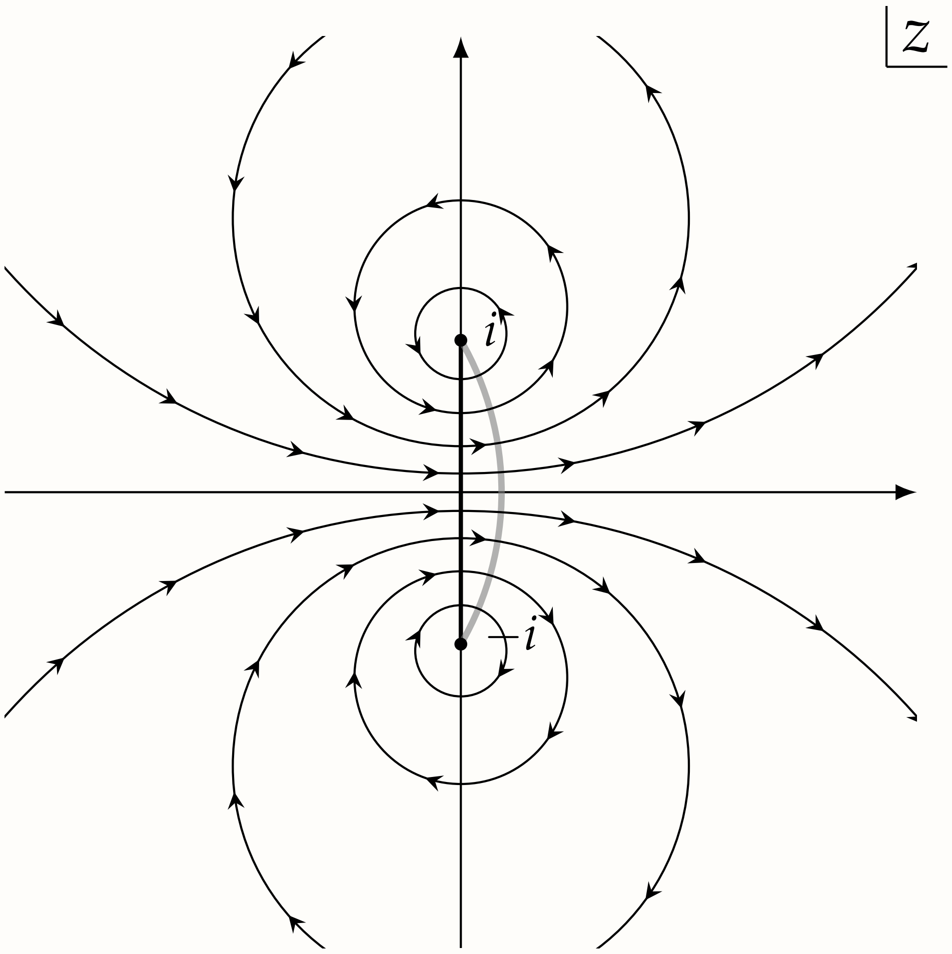
$$g(z) = z^2 + 1$$

$$-\frac{\partial}{\partial t} = -g(z)\frac{\partial}{\partial z} - g(\bar{z})\frac{\partial}{\partial \bar{z}}$$

Entanglement

TT, arXiv: 1904.12414

$$-\frac{\partial}{\partial t} = -z^2\frac{\partial}{\partial z} - \frac{\partial}{\partial z} - \bar{z}^2\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial \bar{z}}$$



Entanglement

TT, arXiv: 1904.12414

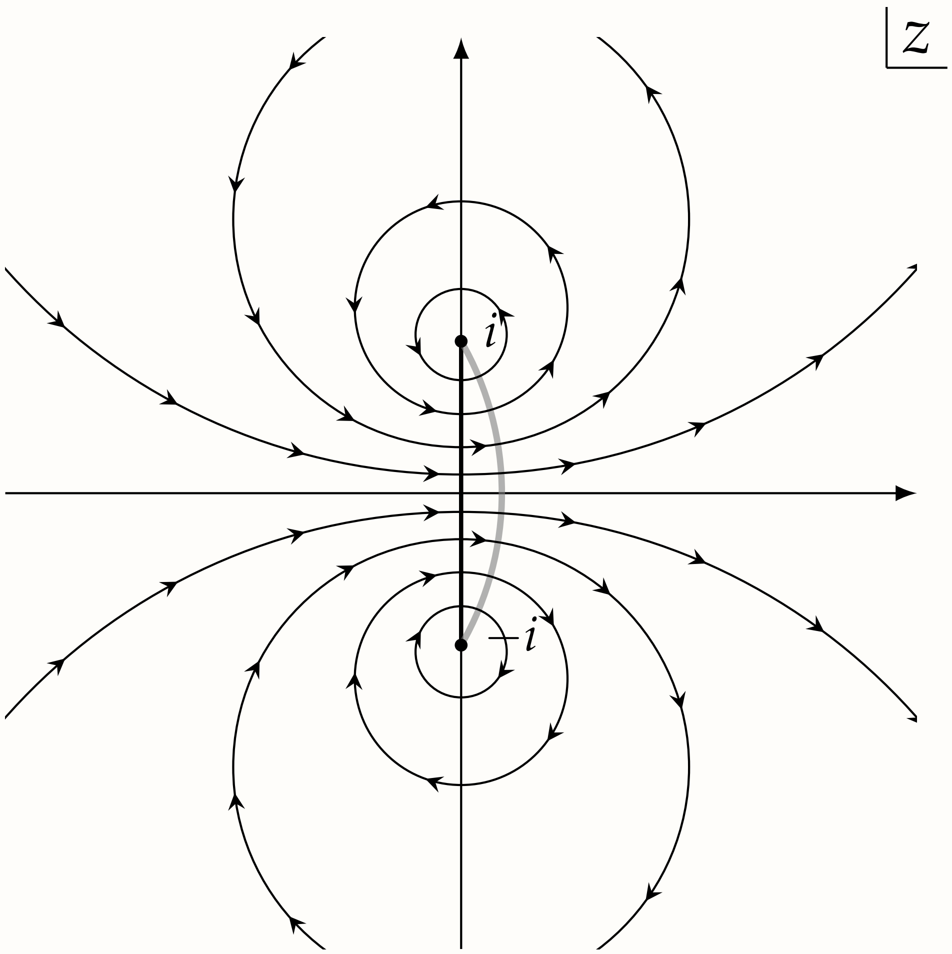
$$\boxed{t + is = \arctan z}$$

$$f_\kappa = \exp\left(\kappa \int^z \frac{dz}{z^2+1}\right)$$

$$= \left(\frac{i+z}{i-z}\right)^{-\frac{i}{2}\kappa} = e^{\kappa \arctan z}$$

$$\mathfrak{l}_\kappa = -(z^2+1)e^{\kappa \arctan z}\frac{\partial}{\partial z}$$

$$[\mathfrak{l}_\kappa, \mathfrak{l}_{\kappa'}] = (\kappa' - \kappa) g(z) f_\kappa(z) f_{\kappa'}(z) \frac{\partial}{\partial z} = (\kappa - \kappa')\mathfrak{l}_{\kappa+\kappa'}$$



Entanglement

TT, arXiv: 1904.12414

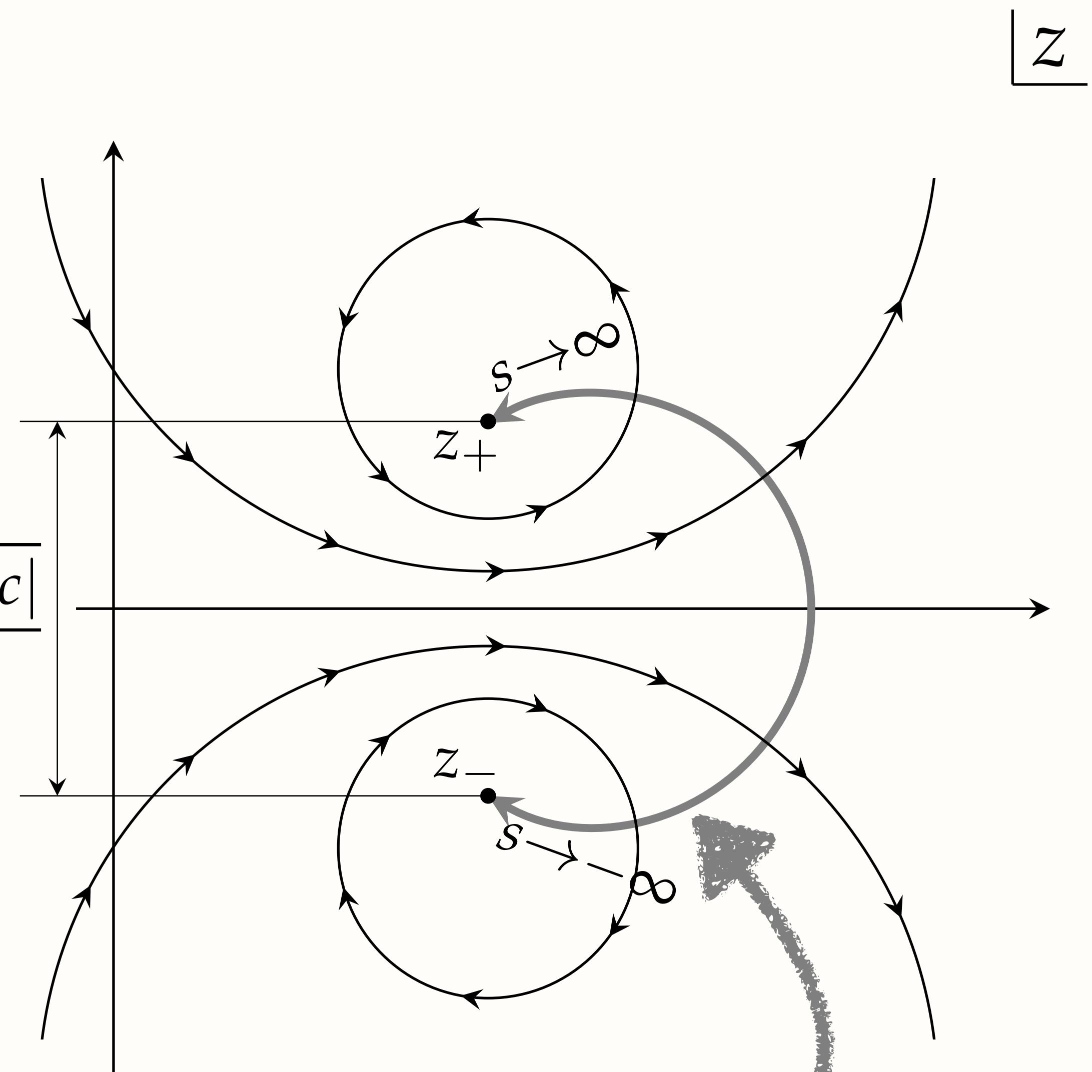
$$g(z) = az^2 + bz + c,$$

$$b^2 - 4ac < 0$$

$$L \equiv \frac{\sqrt{|b^2-4ac|}}{a}$$

$$f_\kappa = \exp(\kappa \int^z \frac{dz}{g(z)})$$

$$\mathcal{L}_\kappa \equiv \int_C dz g(z) f_\kappa(z) T(z)$$



Entanglement

TT, arXiv: 1904.12414

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$

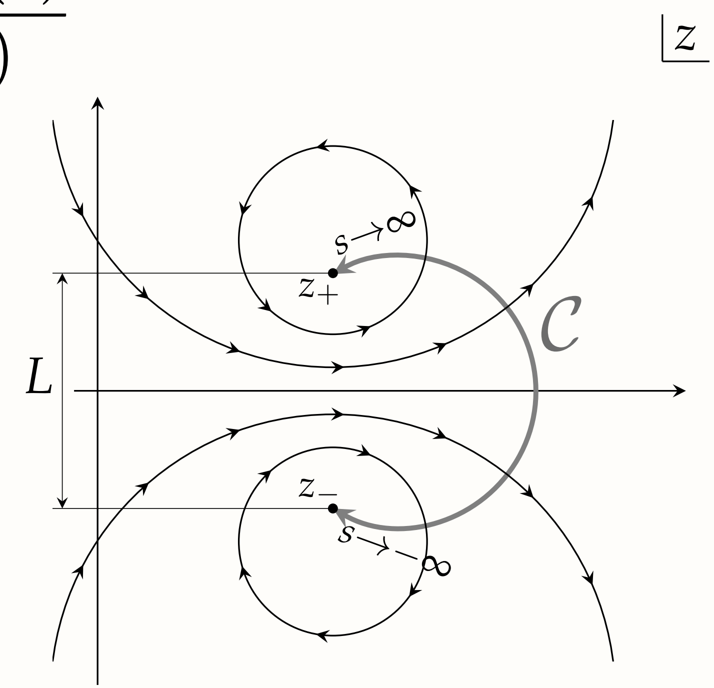
$$\mathrm{CI}[\kappa|\kappa'] \equiv \int_{\mathcal{C}} \frac{dz}{2\pi i}\left\{ g\frac{\partial^3 g}{\partial z^3} + \kappa\left(2\frac{\partial^2 g}{\partial z^2} - \frac{1}{g}\left(\frac{\partial g}{\partial z}\right)^2\right) + \frac{\kappa^3}{g}\right\} f_{\kappa+\kappa'}(z)$$

$$= \left(-(b^2 - 4ac)\kappa + \kappa^3\right)\int_{\mathcal{C}} \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)}$$

Jacobi identity

$$\mathrm{CI}[\kappa|\kappa'] = 0$$

$$\kappa + \kappa' \neq 0$$



Entanglement

TT, arXiv: 1904.12414

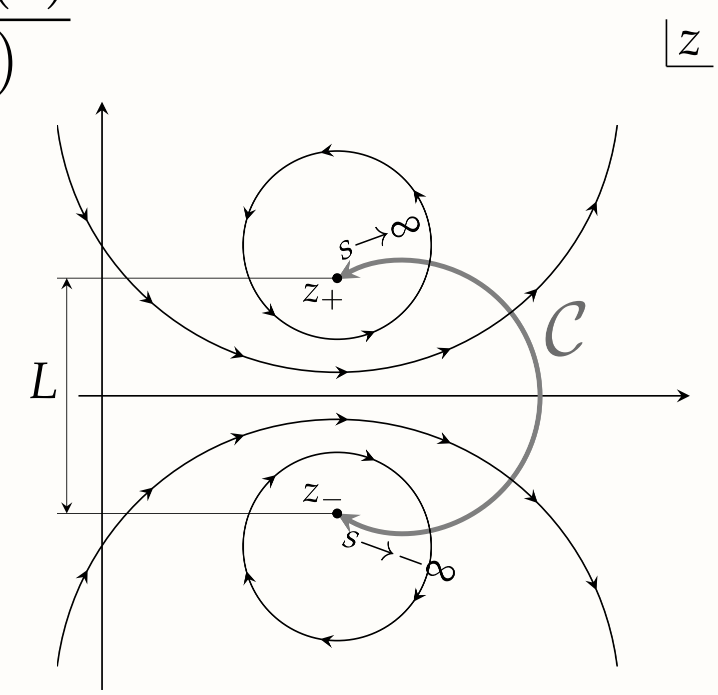
$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$

$$\mathrm{CI}[\kappa|\kappa'] \equiv \int_{\mathcal{C}} \frac{dz}{2\pi i}\left\{g\frac{\partial^3 g}{\partial z^3} + \kappa\left(2\frac{\partial^2 g}{\partial z^2} - \frac{1}{g}\left(\frac{\partial g}{\partial z}\right)^2\right) + \frac{\kappa^3}{g}\right\} f_{\kappa+\kappa'}(z)$$

$$= \left(-(b^2-4ac)\kappa + \kappa^3\right)\int_{\mathcal{C}} \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)}$$

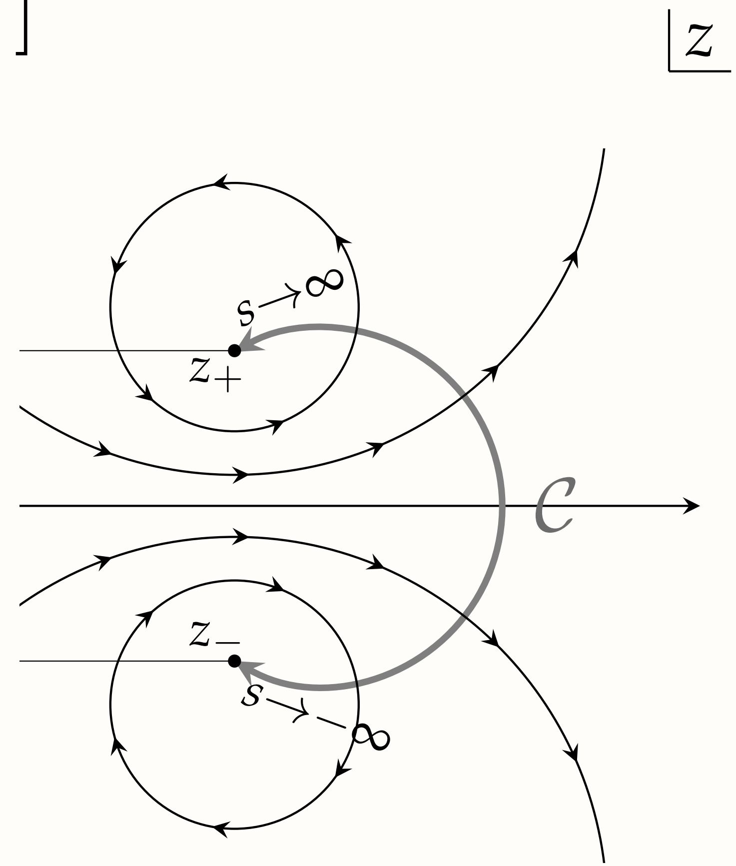
$$\int_{\mathcal{C}} \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)} = \int_{\mathcal{C}} \frac{df_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}$$

$$= \left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial\mathcal{C}}$$



Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$



$$\left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial\mathcal{C}}$$

$$= \frac{1}{2\pi i(\kappa+\kappa')}\left(f_{\kappa+\kappa'}(z_+) - f_{\kappa+\kappa'}(z_-)\right)$$

$$f_\kappa(z) = \exp\left(\kappa\int^z \frac{dz}{a(z-z_+)(z-z_-)}\right) = \exp\left(\frac{\kappa}{a(z_+-z_-)}\ln\left(\frac{z-z_+}{z-z_-}\right)\right),$$

Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$

$$\left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial\mathcal{C}}$$

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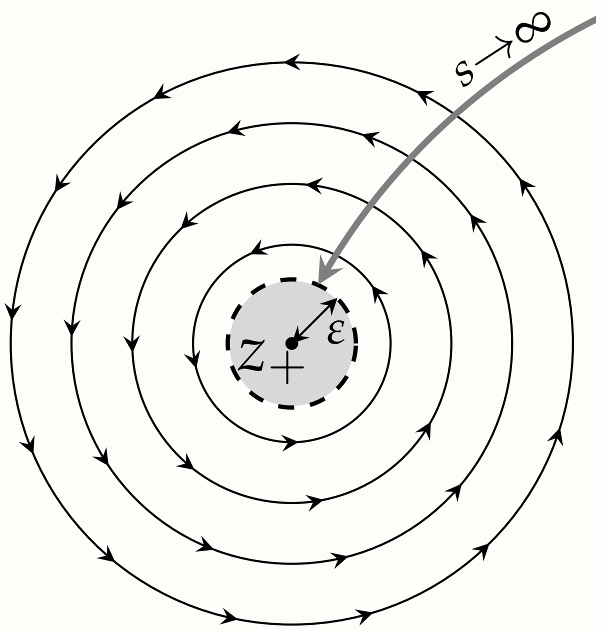
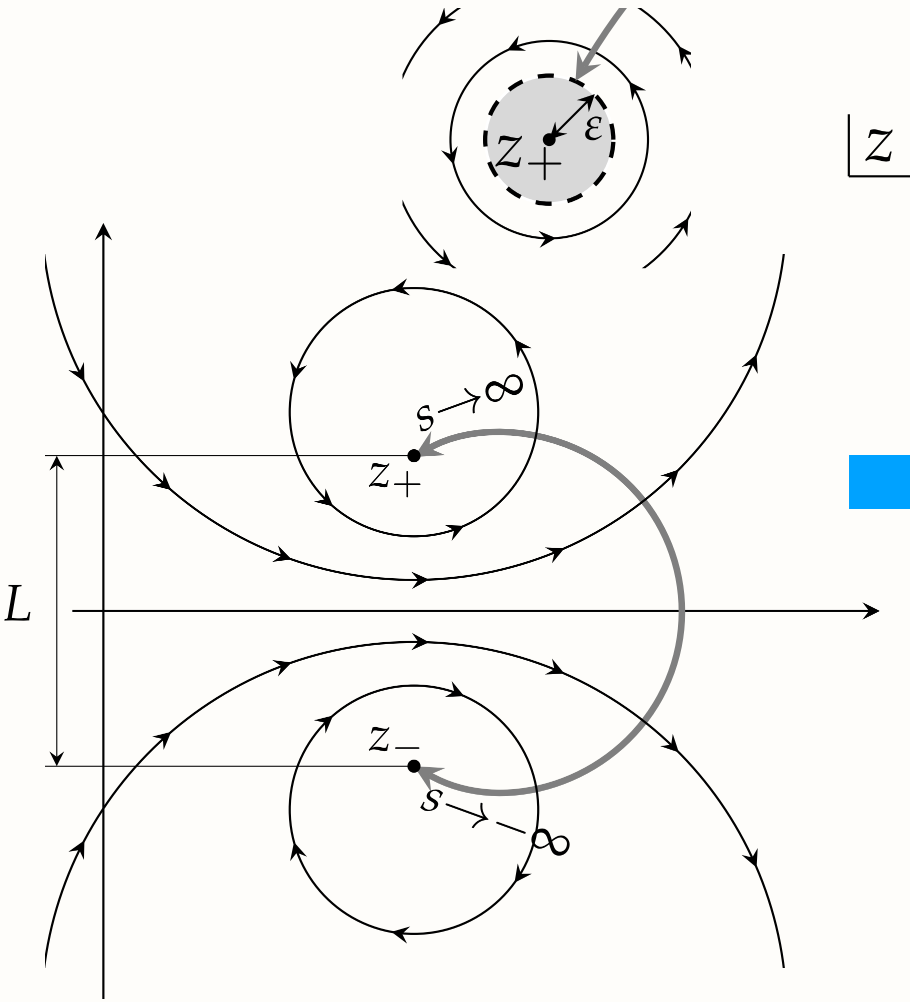
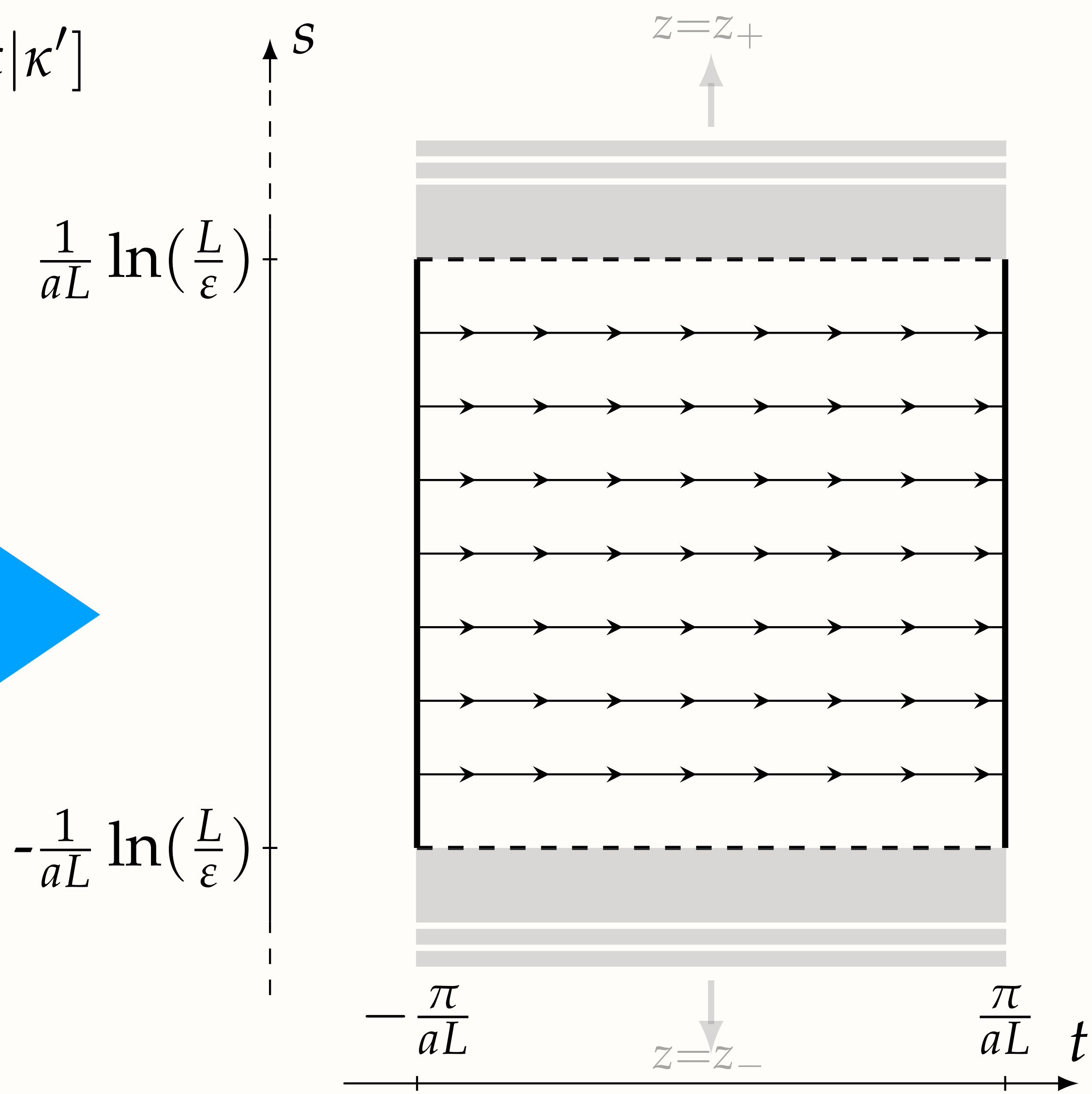


$$f_\kappa(z) = \exp\left(\kappa\int^z \frac{dz}{a(z-z_+)(z-z_-)}\right) = \exp\left(\frac{\kappa}{a(z_+-z_-)}\ln\left(\frac{z-z_+}{z-z_-}\right)\right),$$

Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$



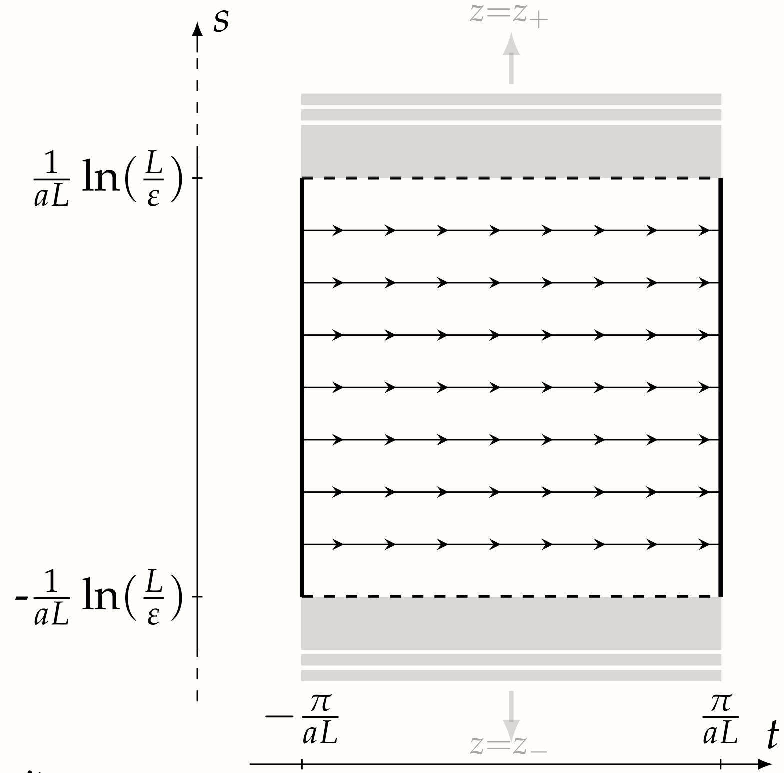


Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$

$$\boxed{2i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right) = 2\pi i n,}$$

$$\kappa = \frac{\pi a L}{\ln(L/\varepsilon)}n, \quad n \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}$$



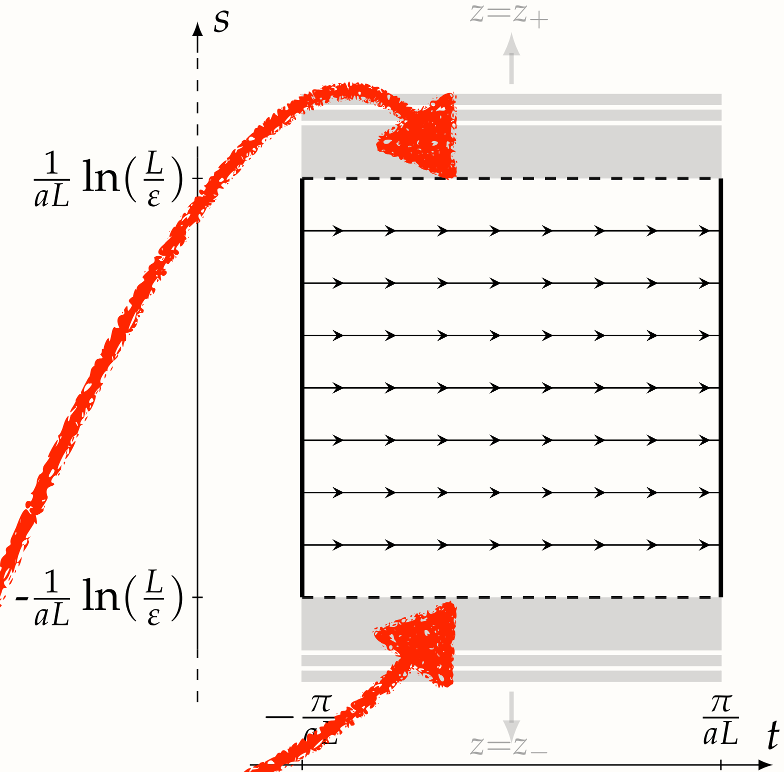
$$\mathrm{CI}[\kappa|\kappa'] = \left(-(b^2-4ac)\kappa + \kappa^3\right)\frac{e^{(\kappa+\kappa')t}}{2\pi i(\kappa+\kappa')}$$
$$\times\left[\exp\left(i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right)\right) - \exp\left(-i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right)\right)\right]$$

Torus geometry

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$

$$\boxed{2i\frac{\kappa + \kappa'}{aL} \ln\left(\frac{L}{\varepsilon}\right) = 2\pi i n,}$$

$$\kappa = \frac{\pi a L}{\ln(L/\varepsilon)} n, \quad n \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}$$



In terms of f_κ, identical

$$\text{CI}[\kappa|\kappa'] = \left(-(b^2 - 4ac)\kappa + \kappa^3\right) \frac{f_{\kappa+\kappa'}}{2\pi i(\kappa + \kappa')}\bigg|_{\partial\mathcal{C}} = 0 \qquad \kappa + \kappa' \neq 0$$

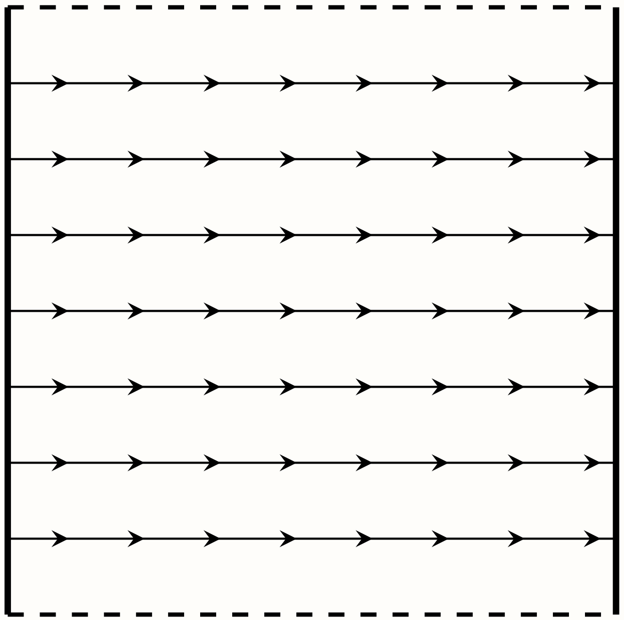
Torus geometry

After rescaling,

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\kappa^3\delta_{\kappa,-\kappa'}$$

Virasoro algebra on

a torus with $\tau = i\frac{\pi}{\ln(L/\varepsilon)}$



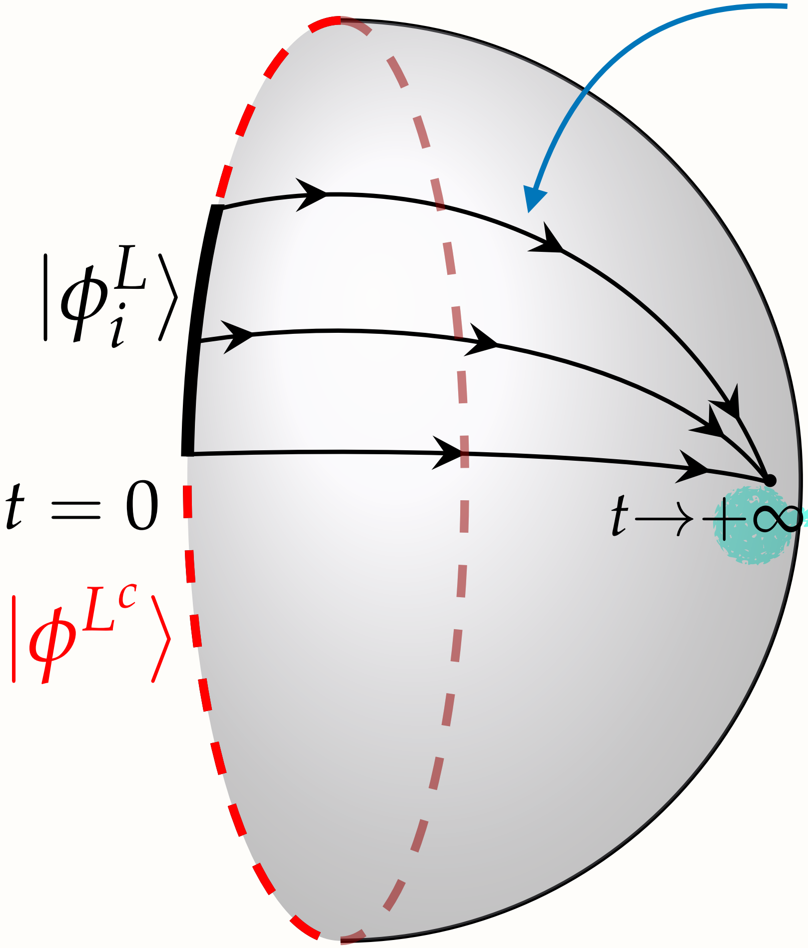
Entanglement Entropy

C. Holzhey, F. Larsen, F. Wilczek

Geometric and Renormalized Entropy in Conformal Field Theory

Nucl. Phys. B 424, 443 (1994)

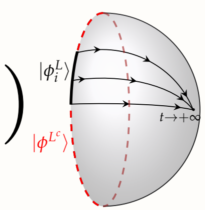
Entanglement Entropy



time flow by

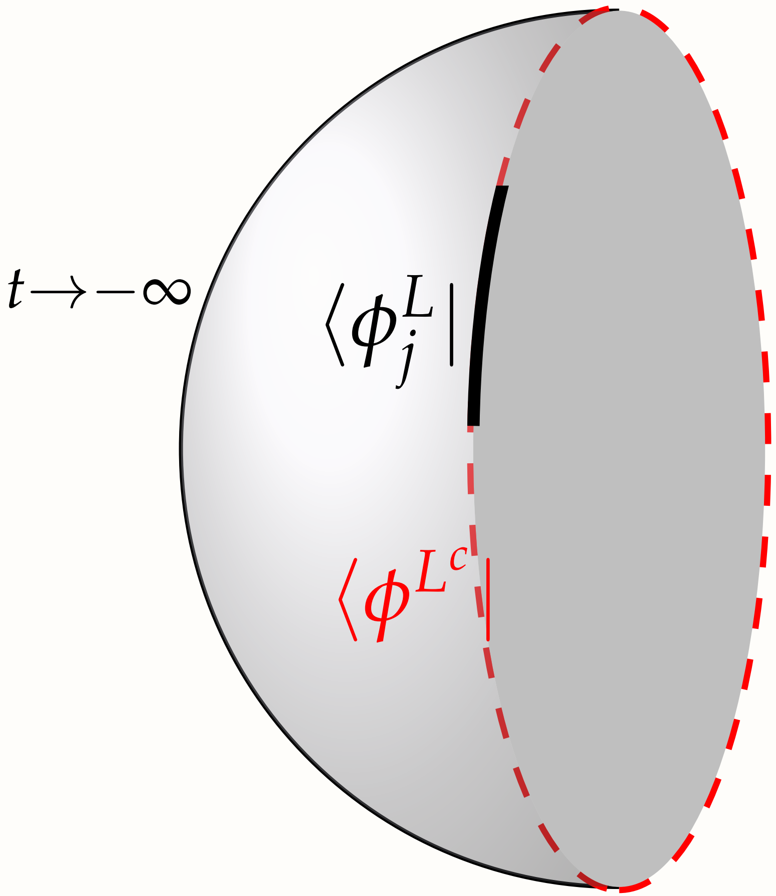
$$e^{-tH}|\phi_i^L\rangle \to |0\rangle$$

$$\langle 0|\phi_i^L\rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

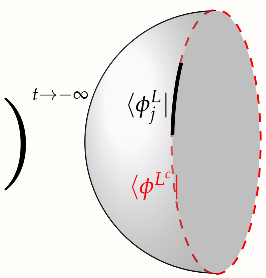


$$\int_{\phi(0,x)=\phi_i^L(x),\phi(0,x^c)=\phi^{L^c}(x^c)} \mathcal{D}\phi(t_{\geq 0}, x)e^{-S}$$

Entanglement Entropy



$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

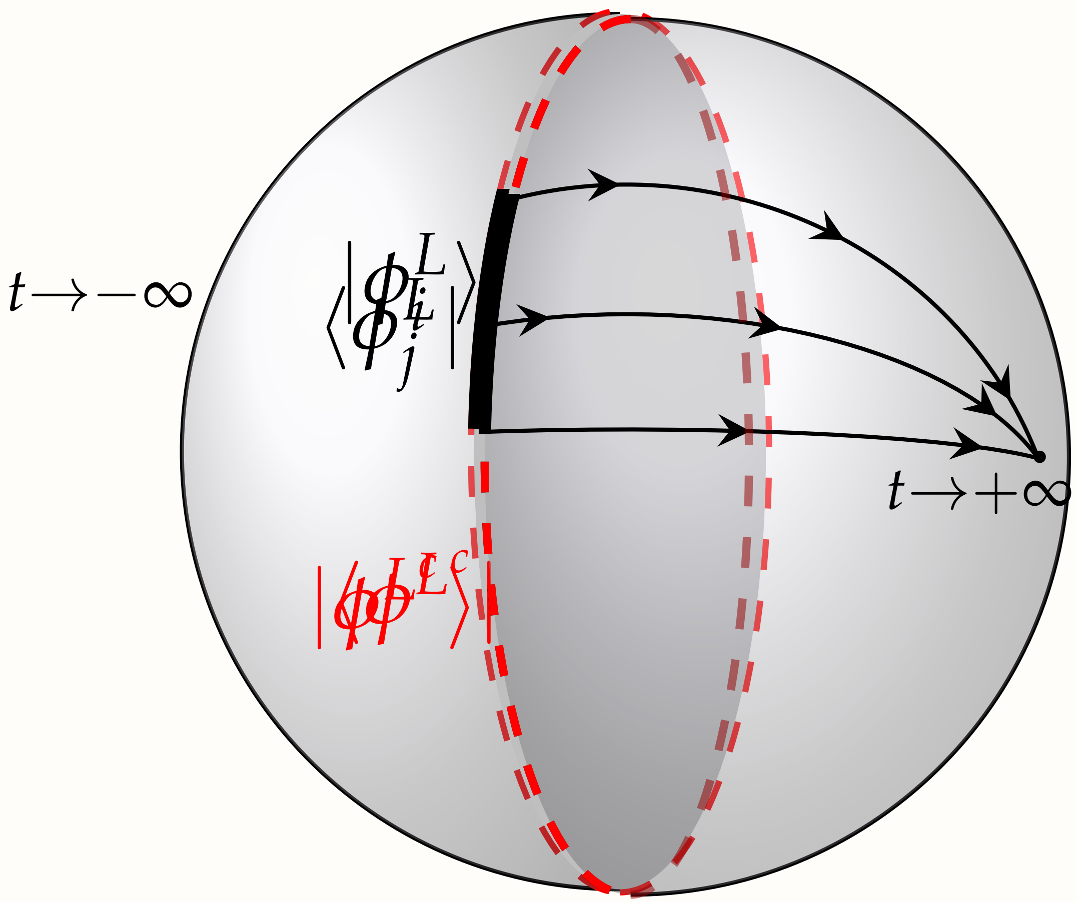


$$\int_{\phi(0,x)=\phi_j^L(x),\,\phi(0,x^c)=\phi^{L^c}(x^c)} \mathcal{D}\phi(t_{\leq 0}, x) e^{-S}$$

Entanglement Entropy

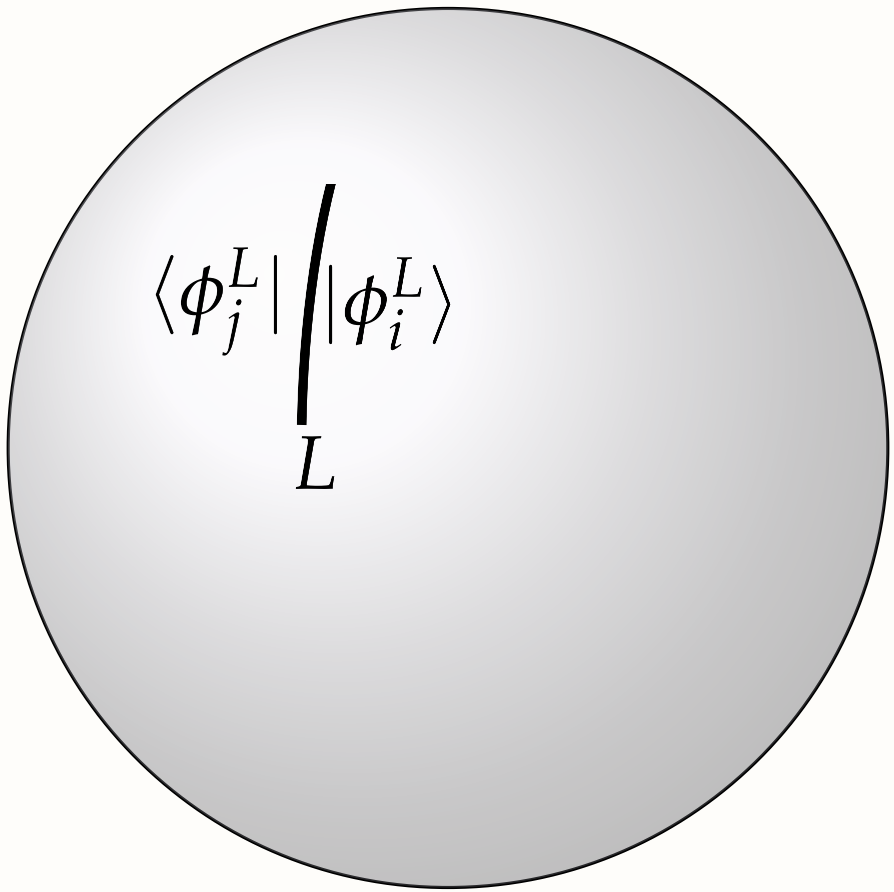
$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$



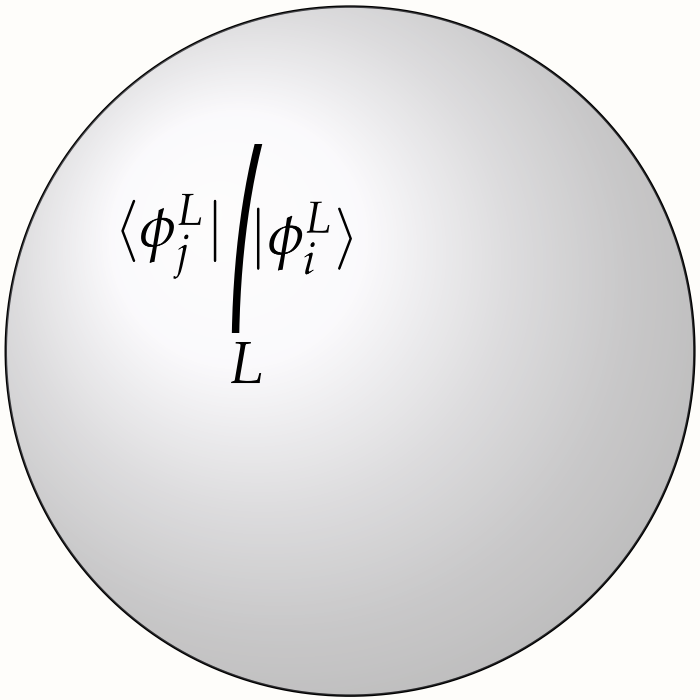
Entanglement Entropy

$$\langle \phi_j^L | 0 \rangle \langle 0 | \phi_i^L \rangle = \int_{\phi(0^-,x)=\phi_j^L(x),\phi(0^+,x)=\phi_i^L(x)} \mathcal{D}\phi(t,x) e^{-S}$$



$$= \left(\mathrm{tr}_{L^c} \left(|0\rangle\langle 0| \right) \right)_{ij}$$

Entanglement Entropy



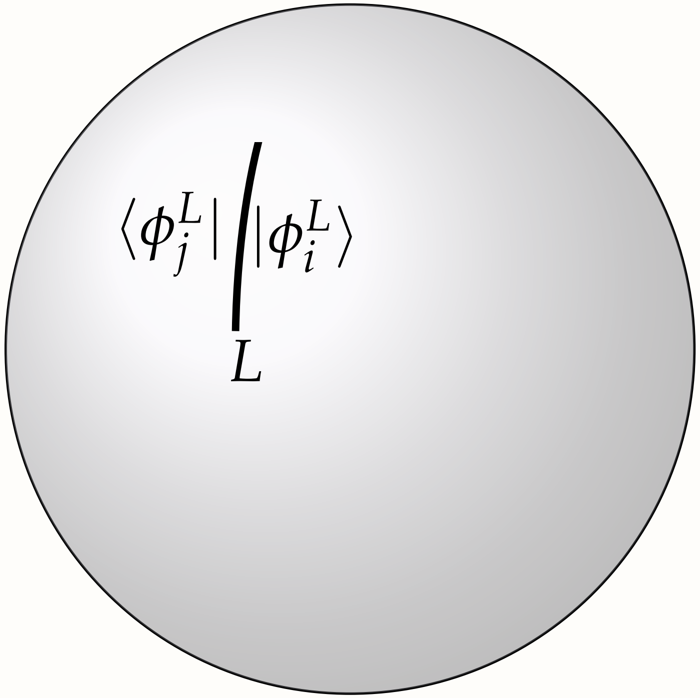
reduced density matrix

$$= \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ij} \equiv Z \times (\rho)_{ij}$$

$$Z = \sum_i \langle \phi_i^L | 0\rangle\langle 0 | \phi_i^L \rangle = \sum_i \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ii}$$

$$\sum_i (\rho)_{ii} = 1$$

Entanglement Entropy



reduced density matrix

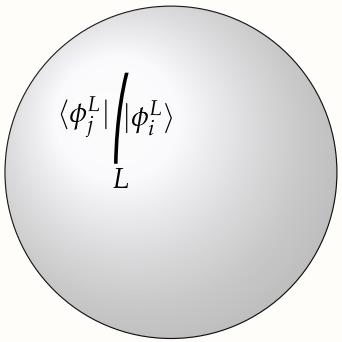
$$= \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ij} \equiv Z \times (\rho)_{ij}$$

$$Z = \sum_i \langle\phi_i^L|0\rangle\langle 0|\phi_i^L\rangle = \sum_i \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ii}$$

$$\sum_i (\rho)_{ii} = 1$$

$$-\mathrm{tr}\left(\rho \ln \rho\right) = S$$

Entanglement Entropy



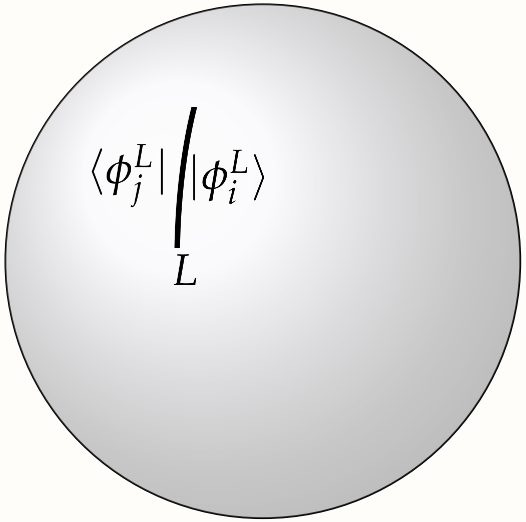
$$\equiv Z \times (\rho)_{ij}$$

$$\rho = \frac{e^{-TH_{\mathrm{mod}}}}{\mathrm{tr}\left(e^{-TH_{\mathrm{mod}}}\right)} = \frac{e^{-TH_{\mathrm{mod}}}}{Z}$$

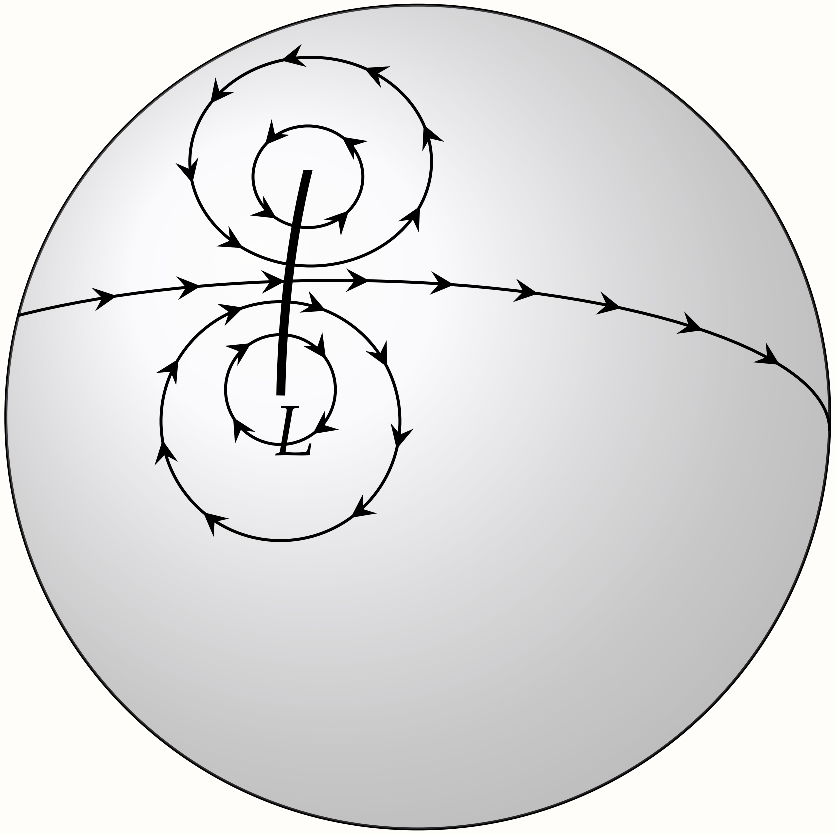
modular (entanglement) Hamiltonian

$$\mathrm{tr} e^{-nTH_{\mathrm{mod}}} = Z^n \mathrm{tr}\rho^n \quad \Rightarrow \quad - \frac{d}{dn} \mathrm{tr}\rho^n \Big|_{n=1} = -\mathrm{tr}\left(\rho \ln \rho\right) = S$$

Entanglement Hamiltonian



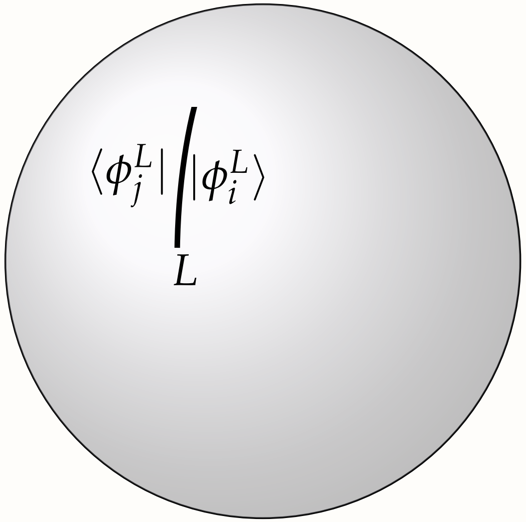
$$\equiv Z \times (\rho)_{ij}$$



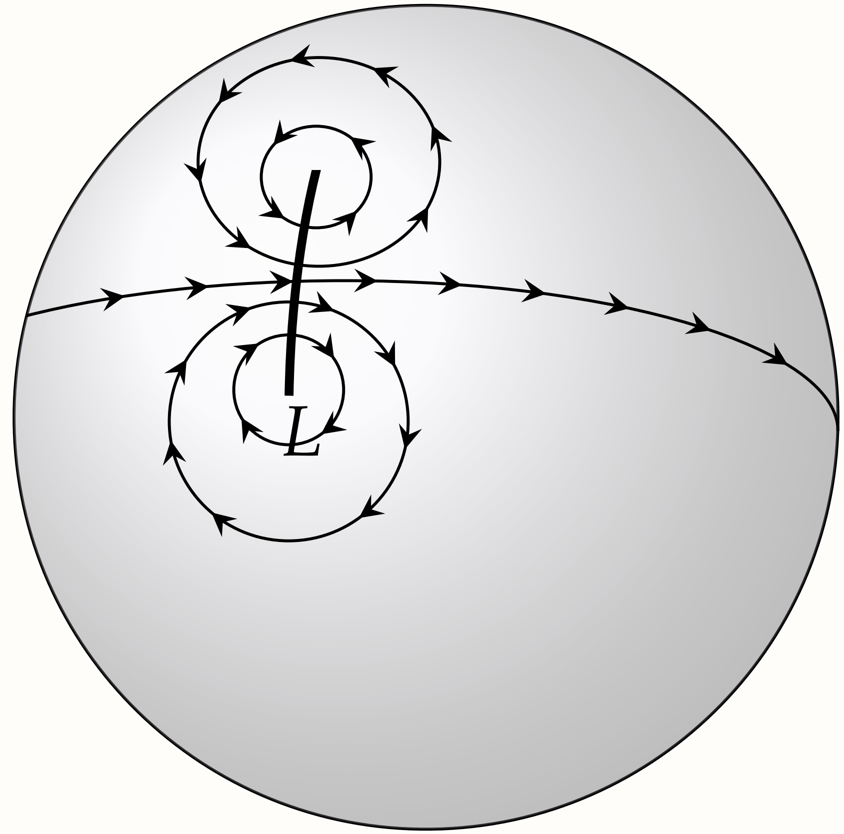
modular (entanglement) Hamiltonian

$$\rho = \frac{e^{-TH_{\text{mod}}}}{\text{tr}\left(e^{-TH_{\text{mod}}}\right)} = \frac{e^{-TH_{\text{mod}}}}{Z}$$

Entanglement Hamiltonian



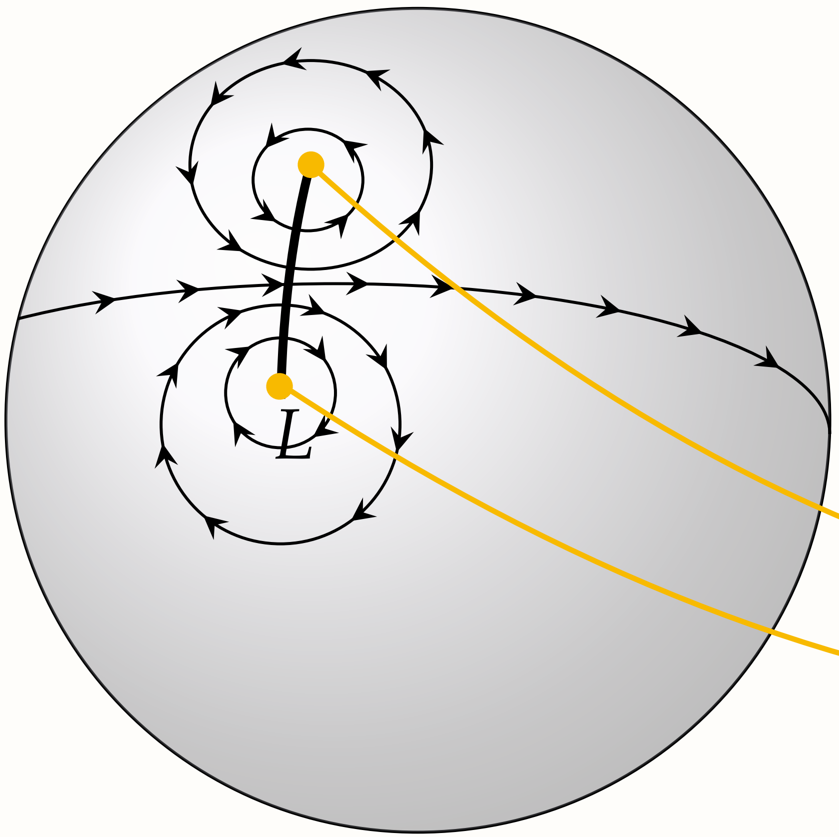
$$\equiv Z \times (\rho)_{ij}$$



modular (entanglement) Hamiltonian

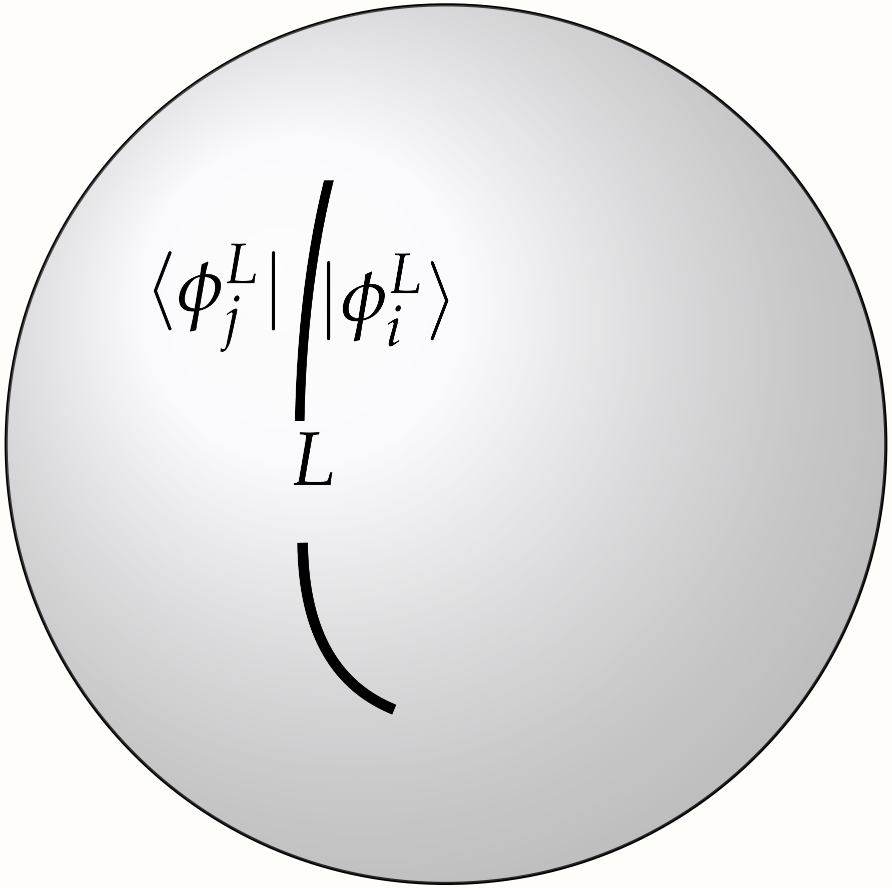
$$H_{\mathrm{mod}} = aL_1 + bL_0 + cL_{-1} + a\bar{L}_1 + b\bar{L}_0 + c\bar{L}_{-1} = \mathcal{L}_0 + \bar{\mathcal{L}}_0$$

Entanglement Hamiltonian



$$S = \frac{c_{\text{CFT}} + \bar{c}_{\text{CFT}}}{6} \ln \frac{L}{\varepsilon}$$

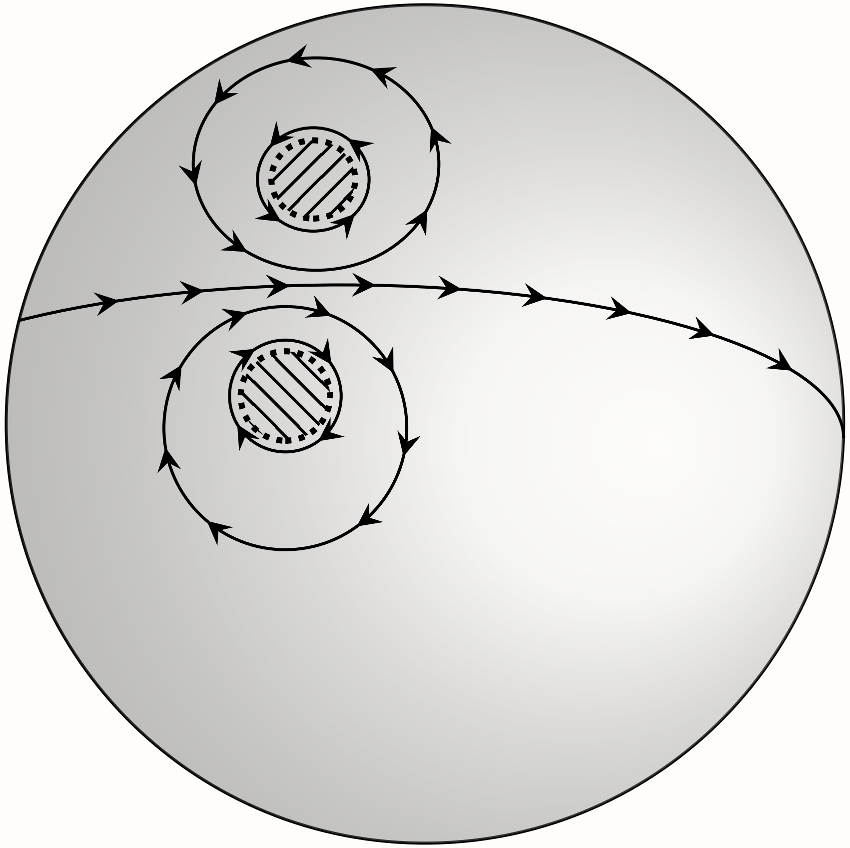
Two or more sections



J. Cardy, E. Tonni, J. Stat. Mech. (2016) 123103

G. Wong, JHEP04(2019)045

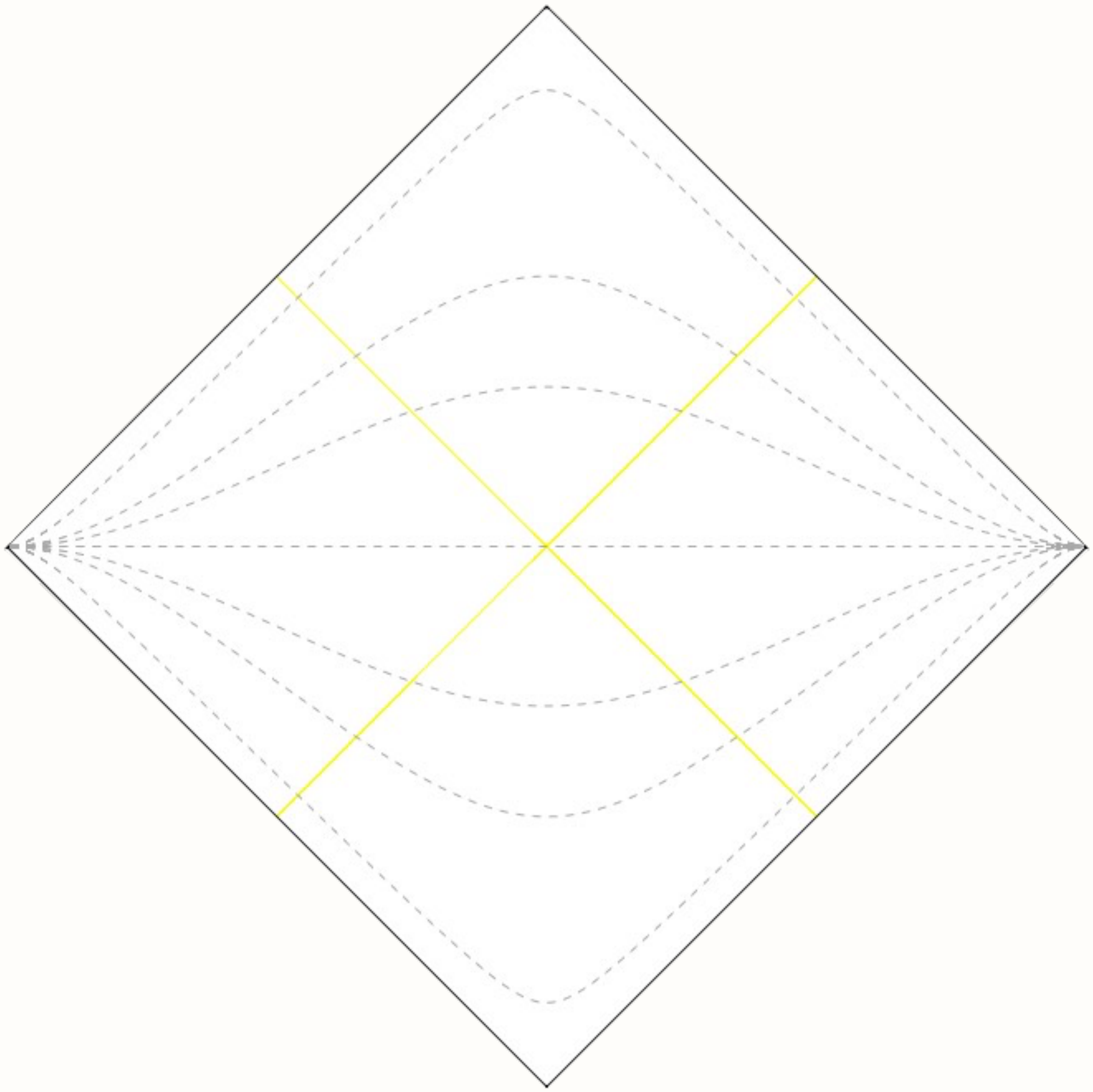
One-loop and higher



larger holes

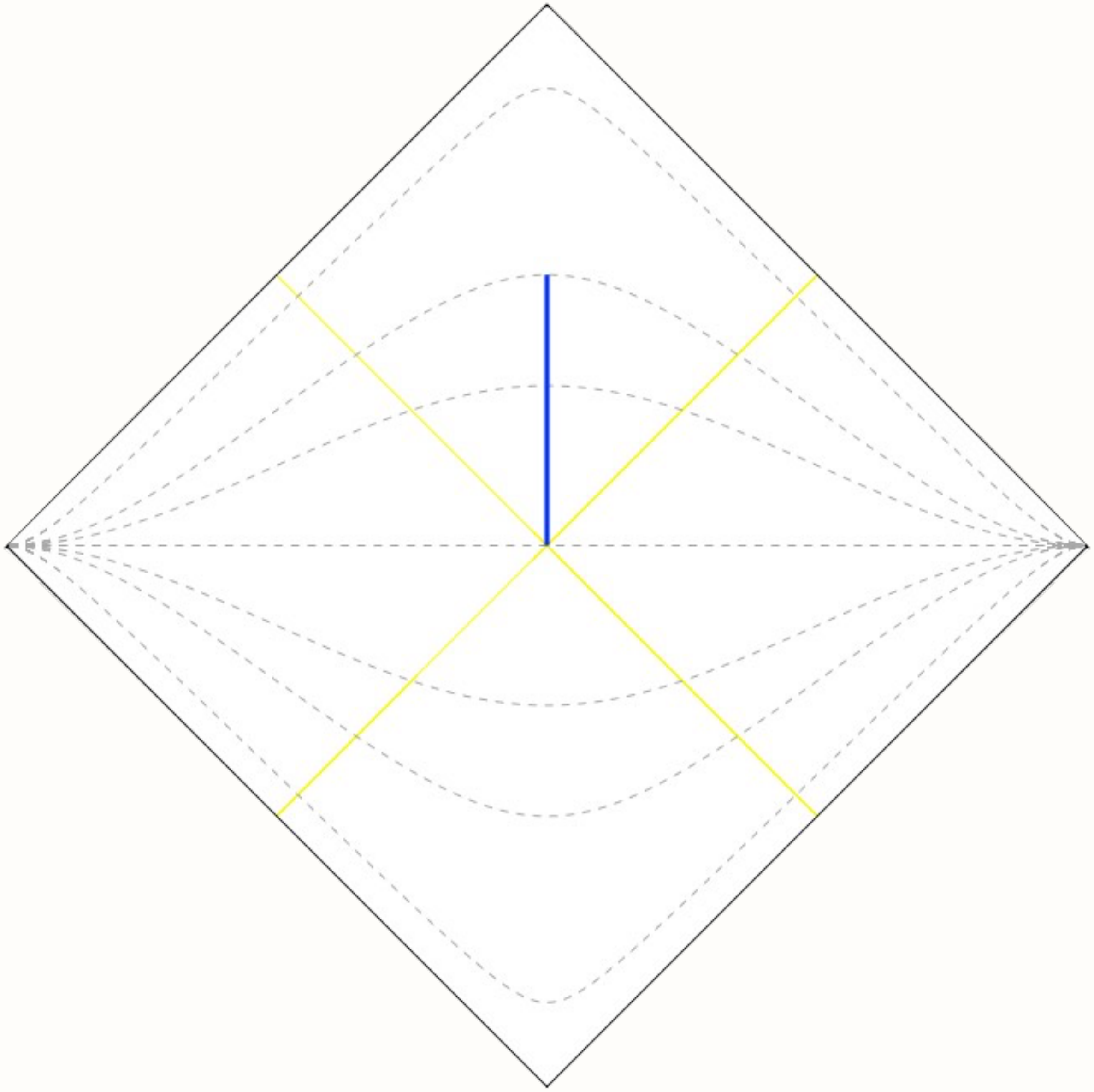
Minkowski CFT

Lüscher, Mack ('75)



Minkowski CFT

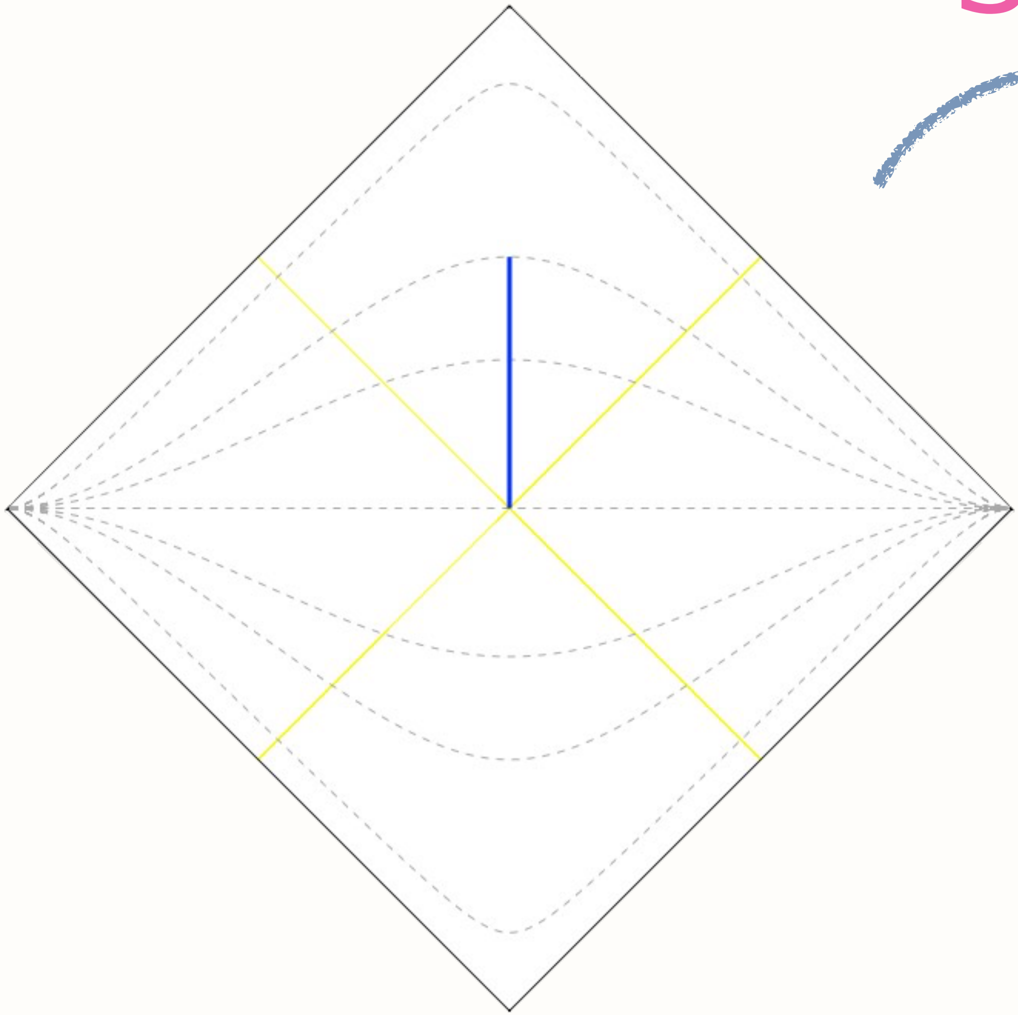
Lüscher, Mack ('75)

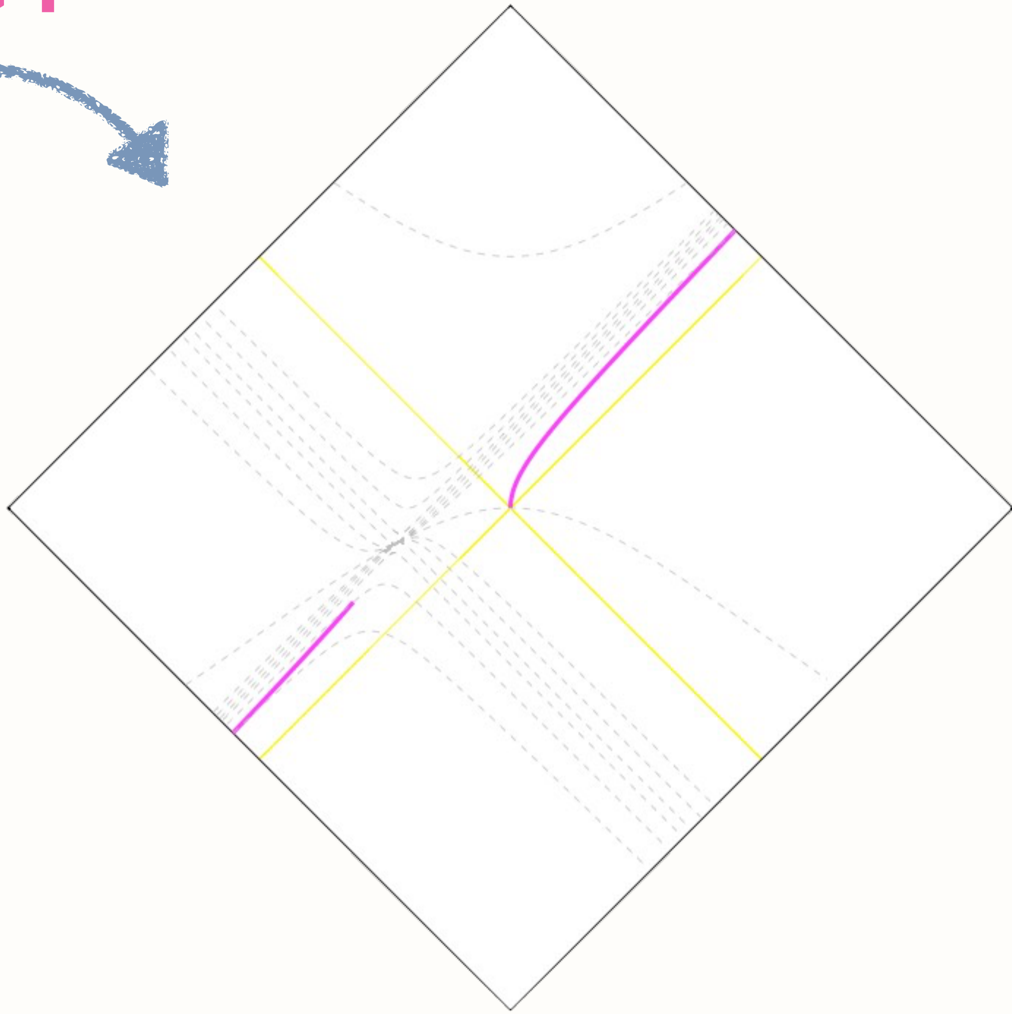


Minkowski CFT

Lüscher, Mack ('75)

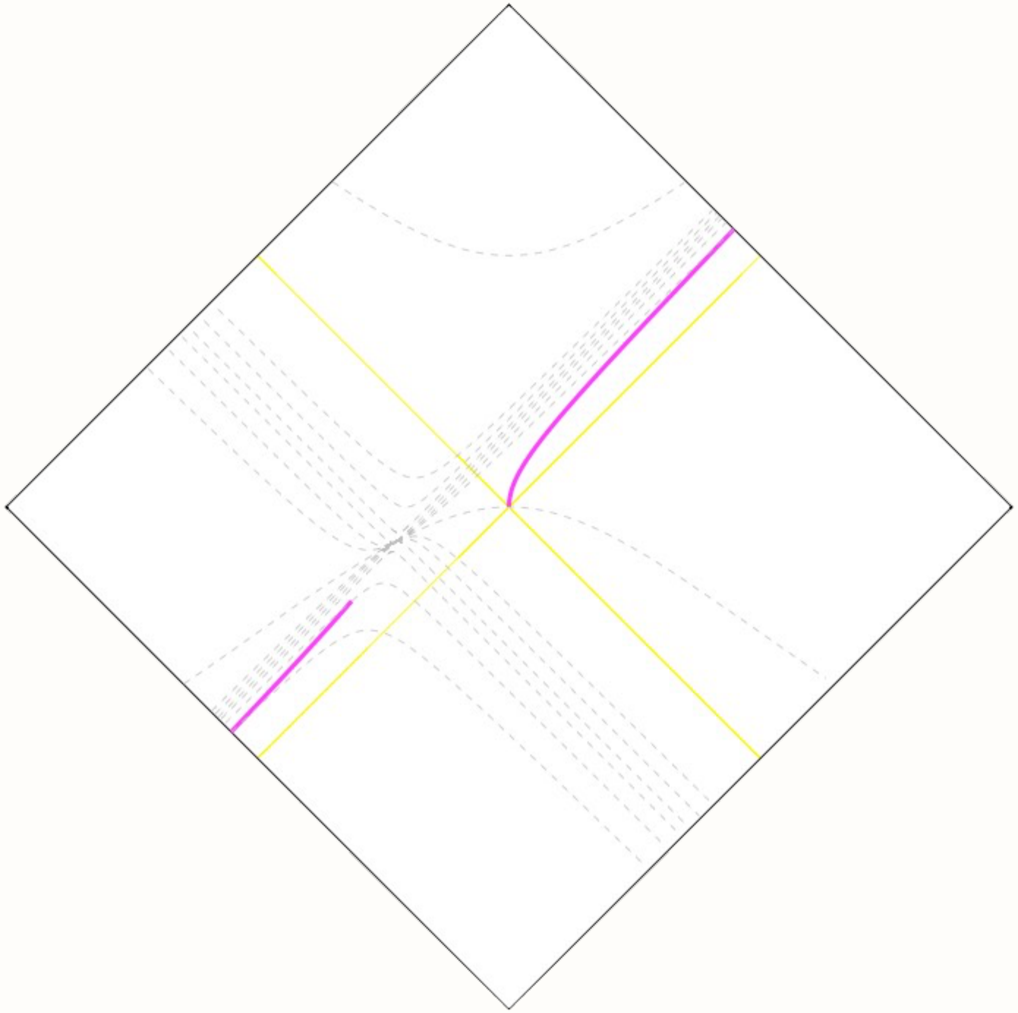
SCT





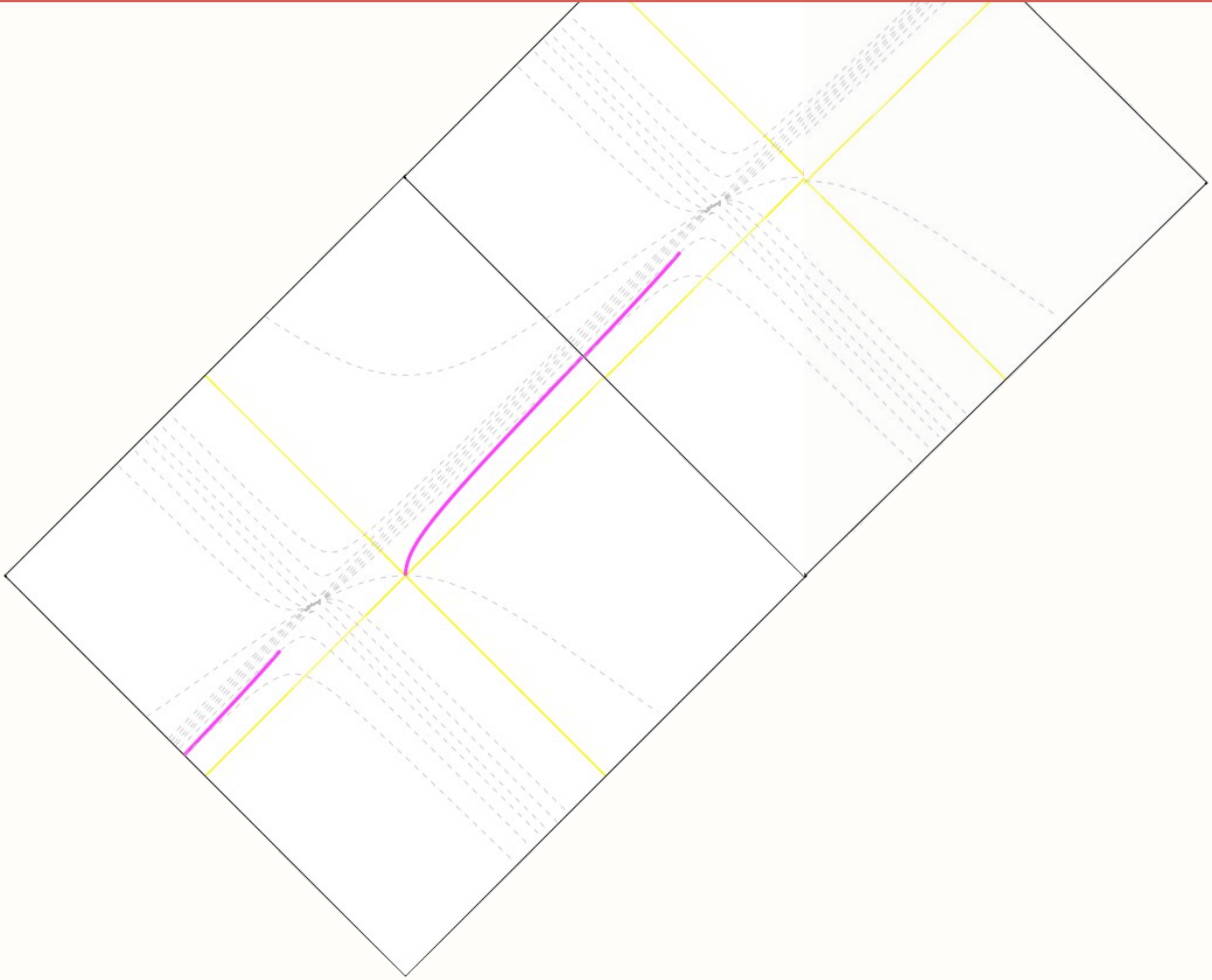
Minkowski CFT

Lüscher, Mack ('75)



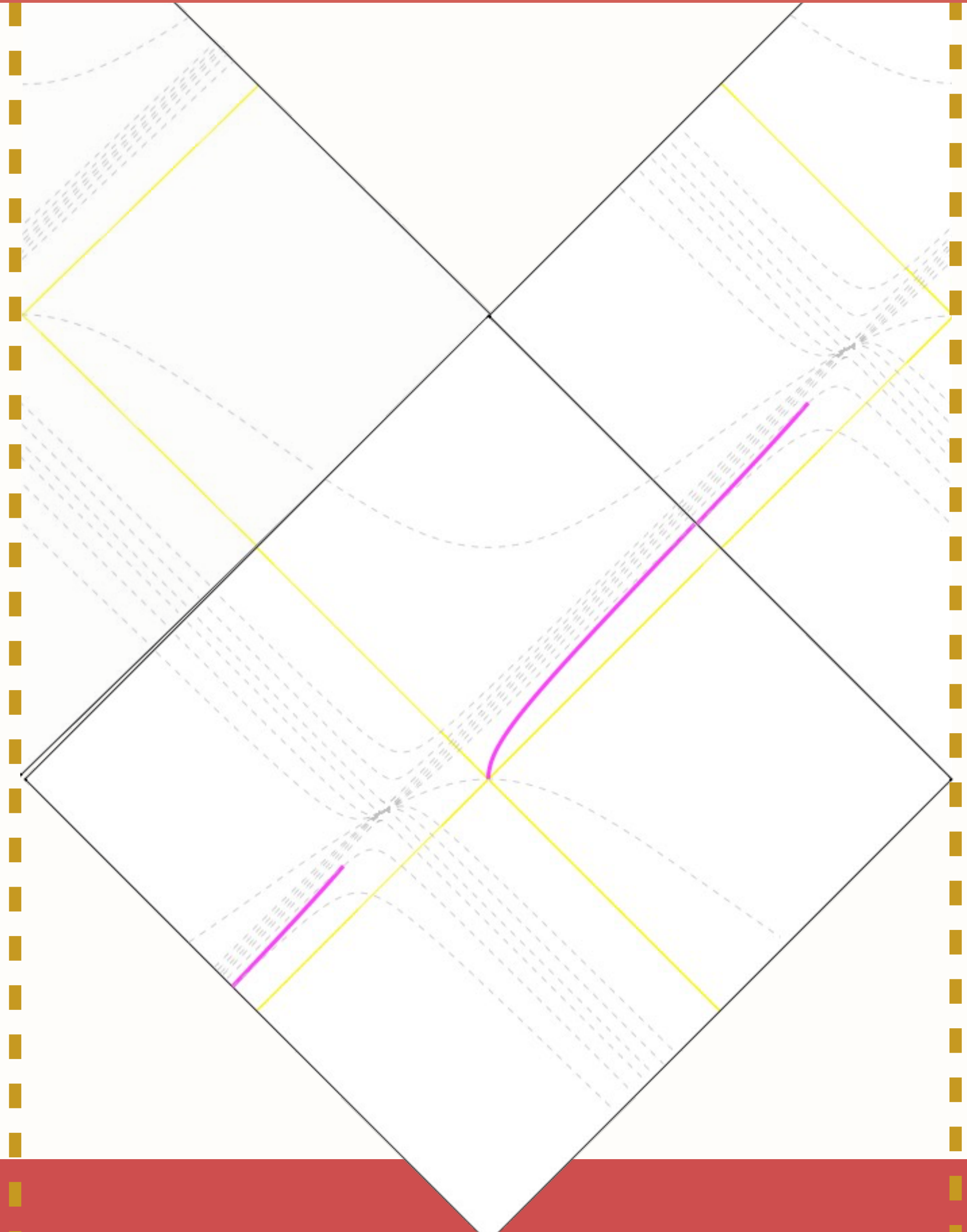
Minkowski CFT

Lüscher, Mack ('75)



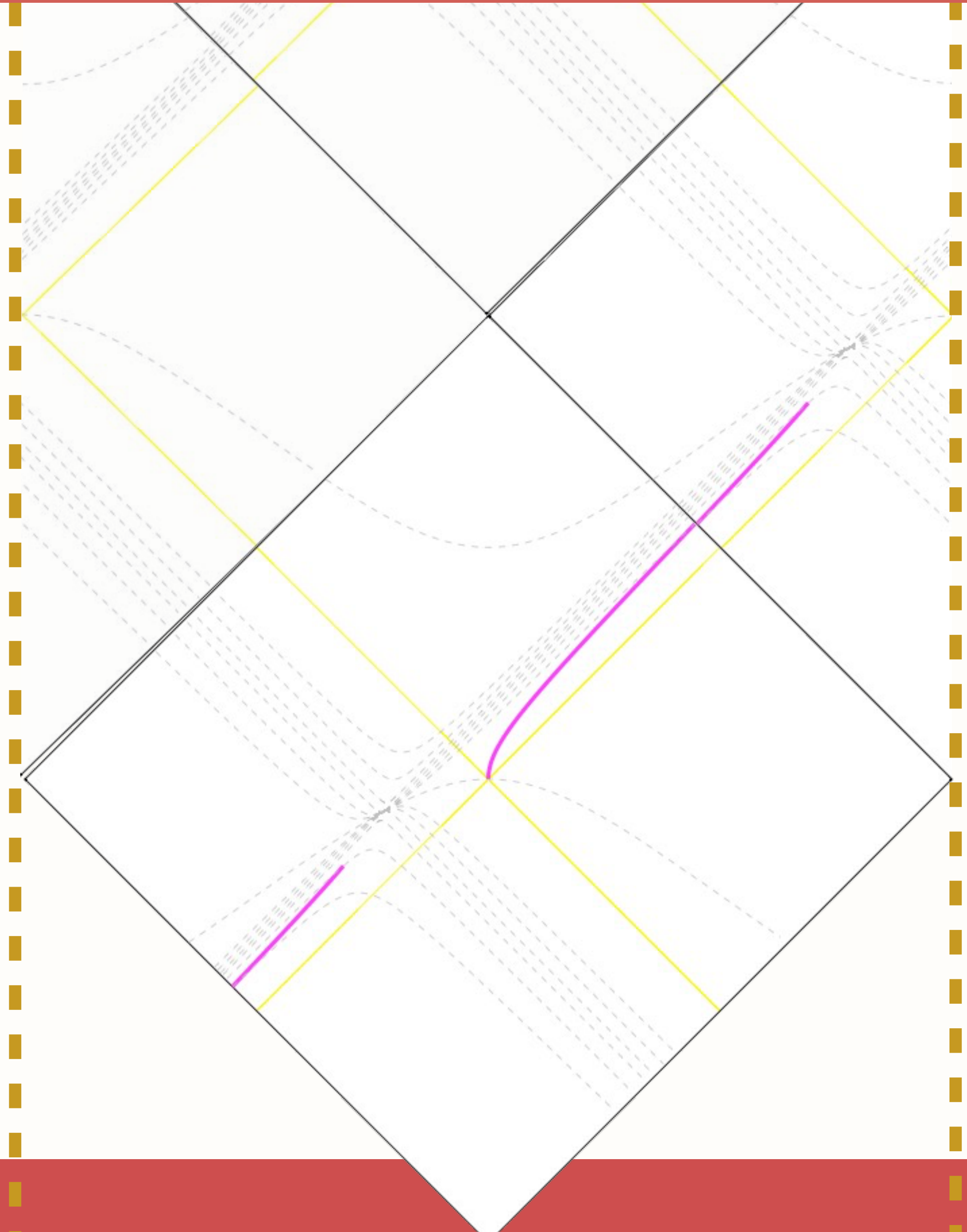
Minkowski CFT

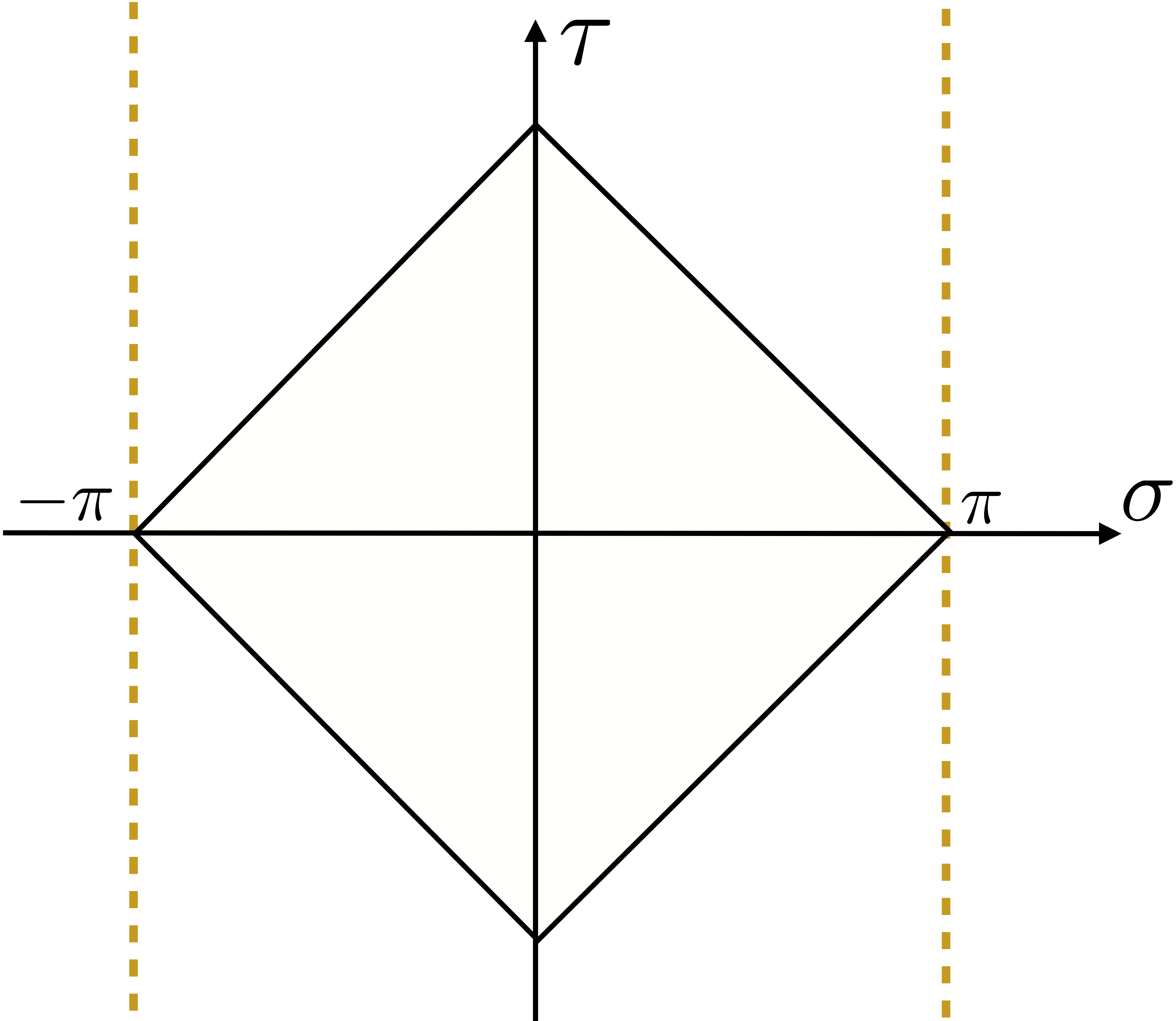
Lüscher, Mack ('75)

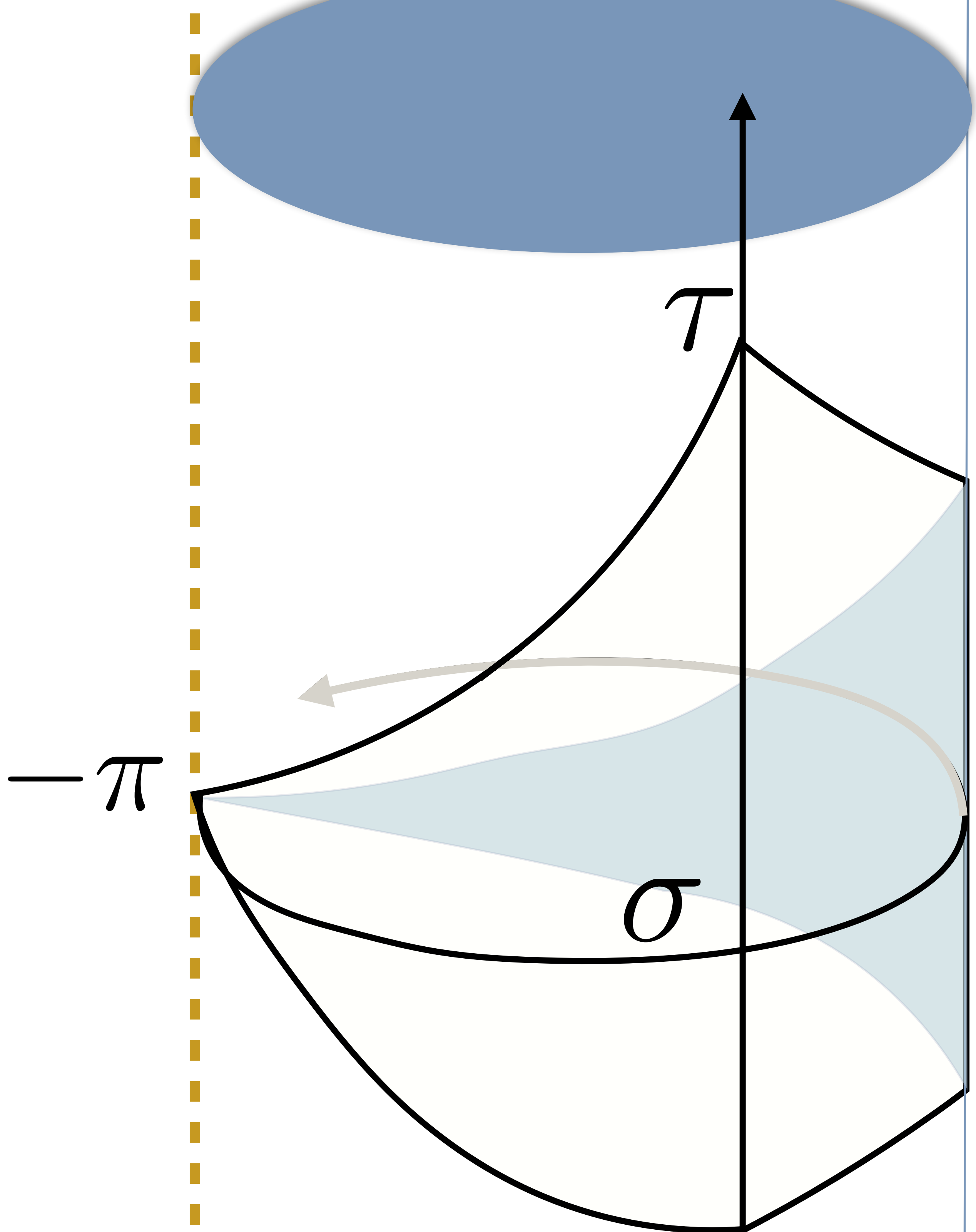


Minkowski CFT

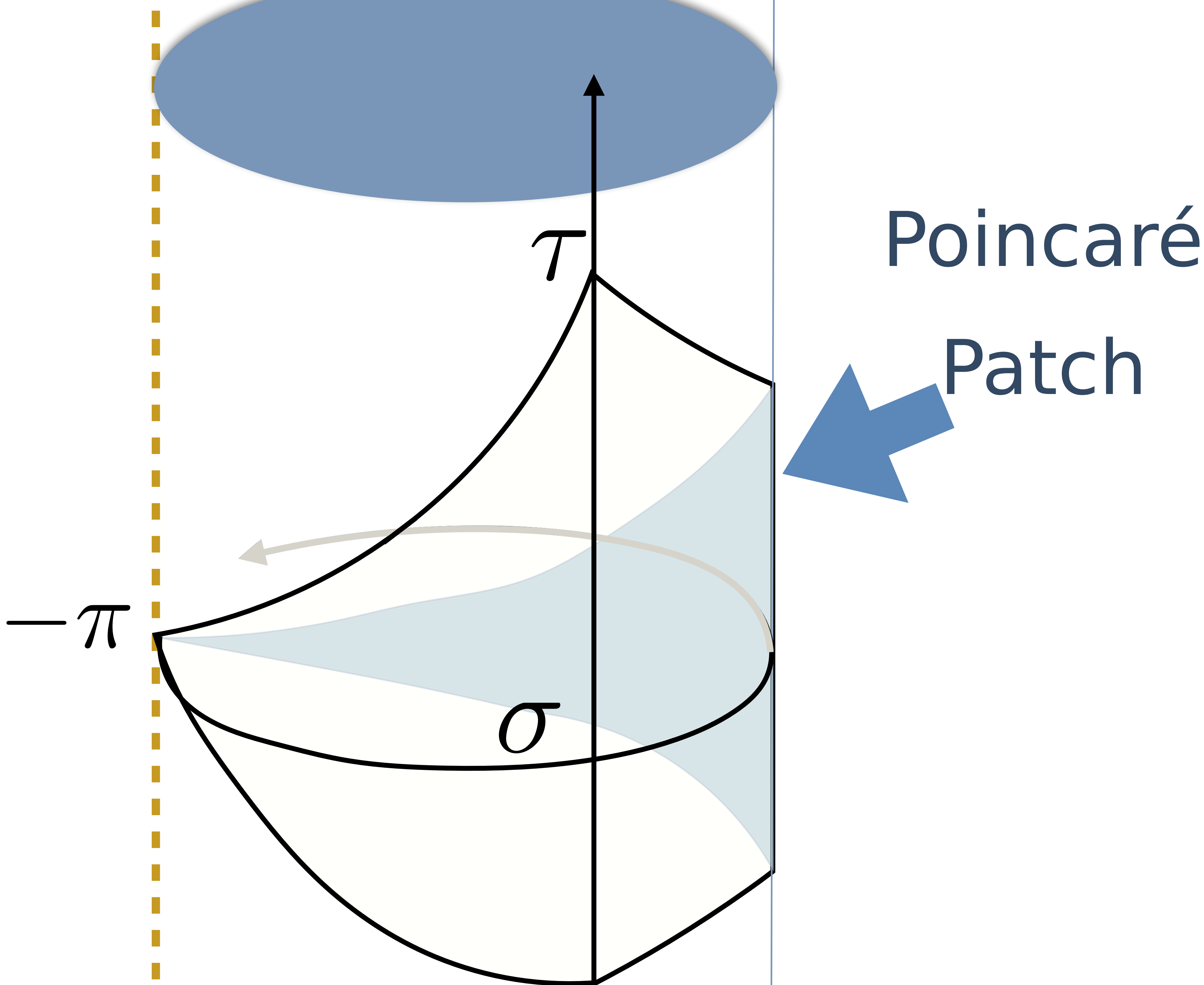
Lüscher, Mack ('75)



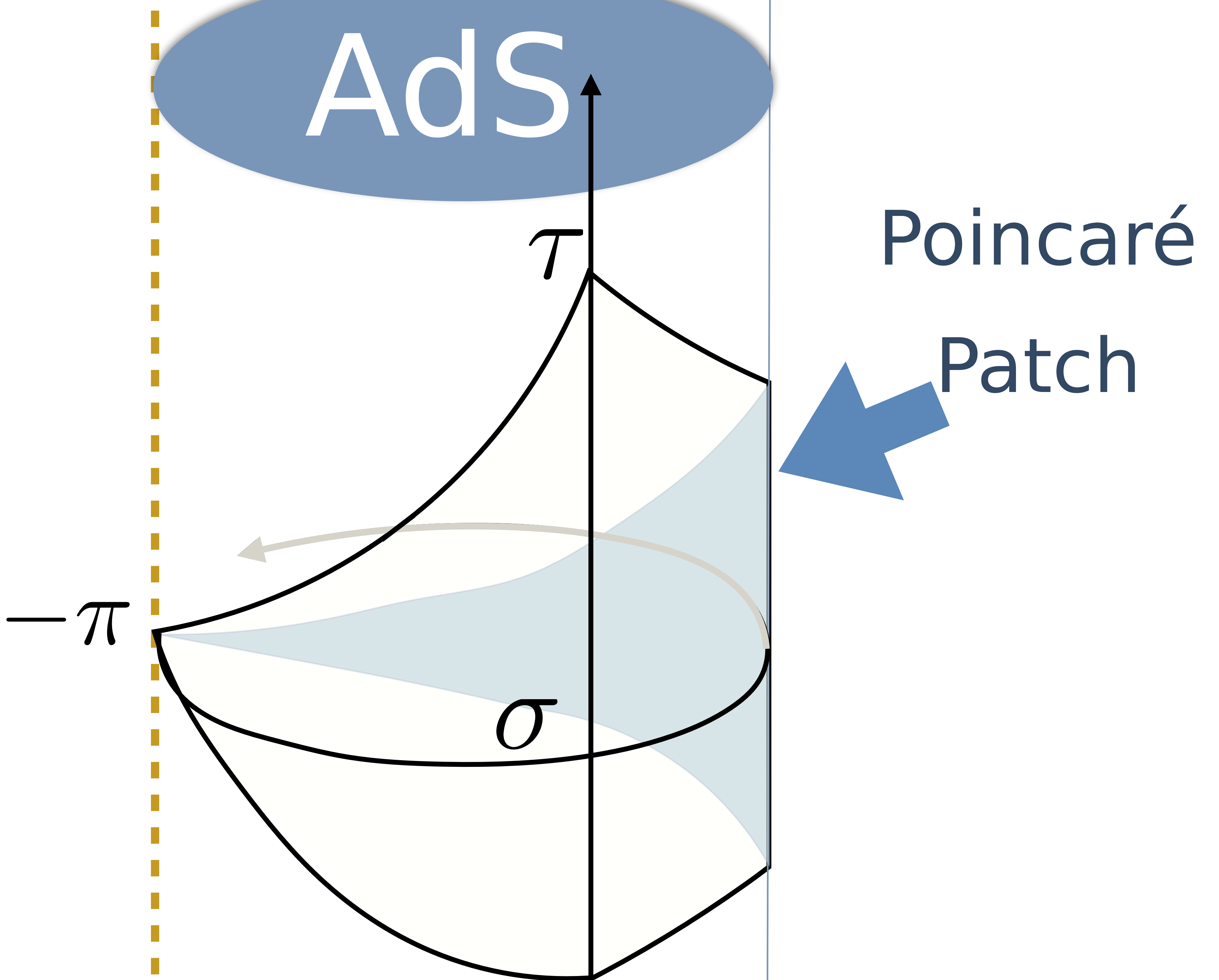

τ
σ
$-\pi$
π



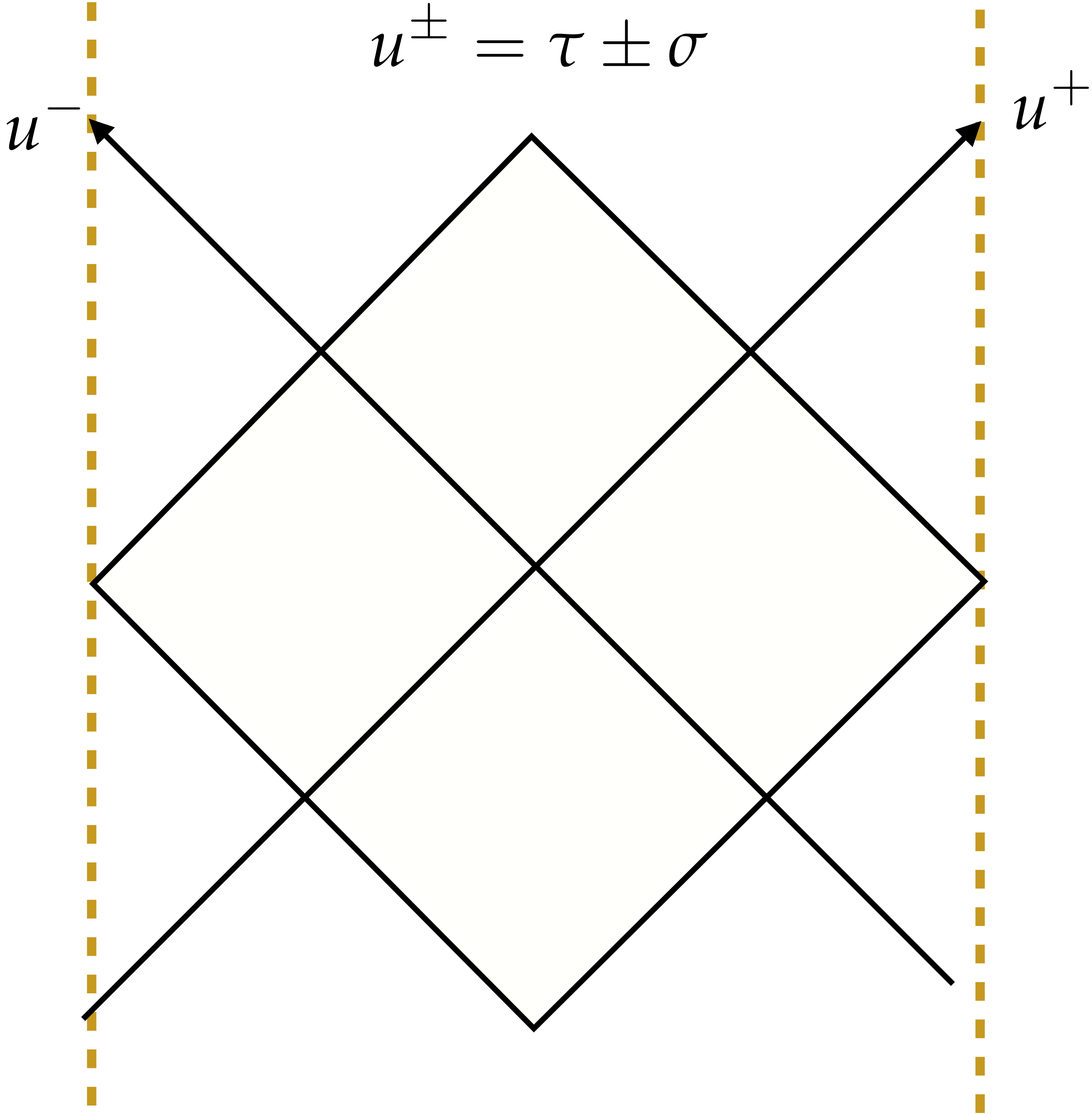
τ
$-\pi$
σ



Poincaré
Patch
τ
$-\pi$
σ

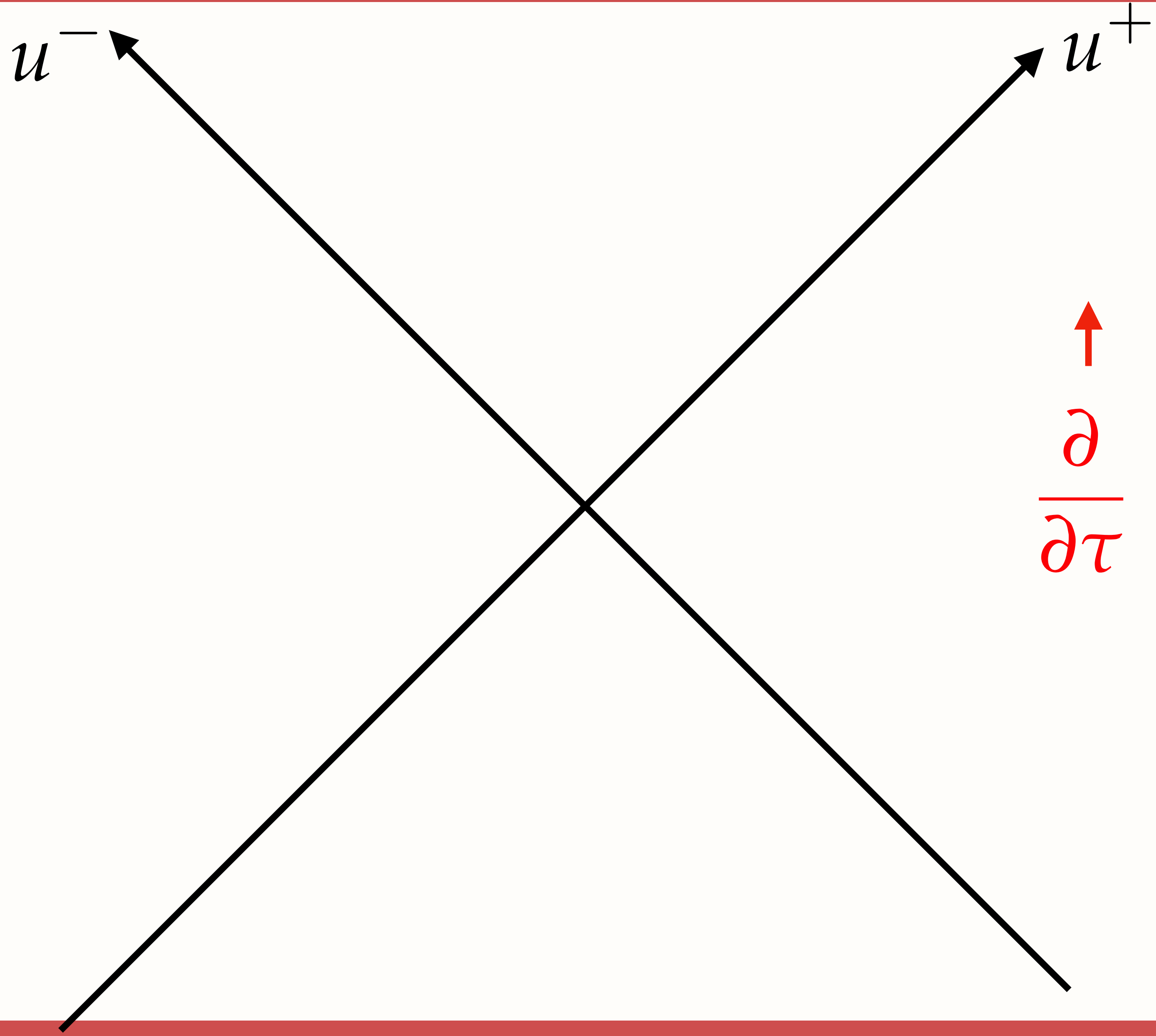


AdS
τ
Poincaré
Patch
$-\pi$
σ



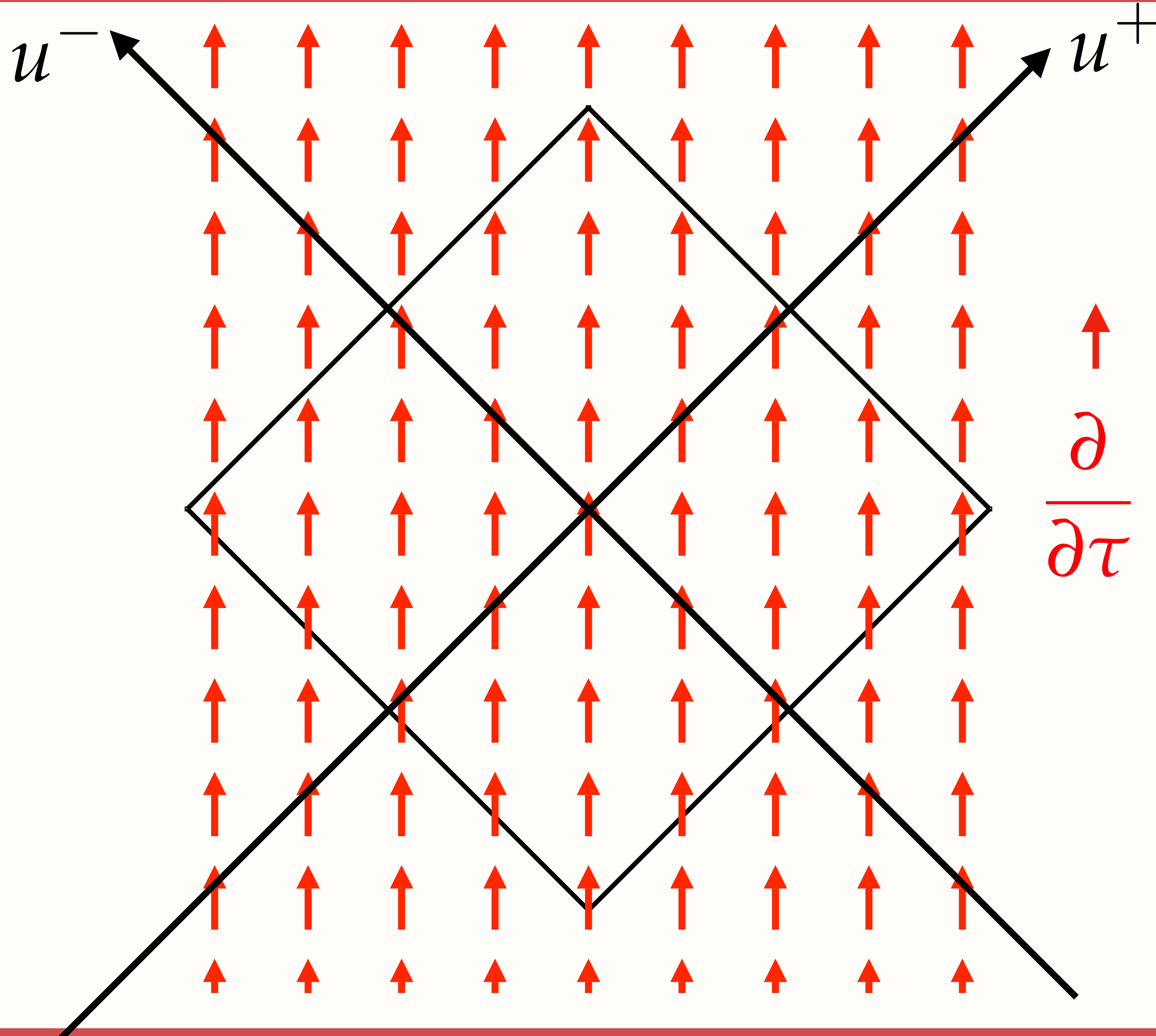
$u^{\pm} = \tau \pm \sigma$
u^{-}
u^{+}

Minkowski CFT



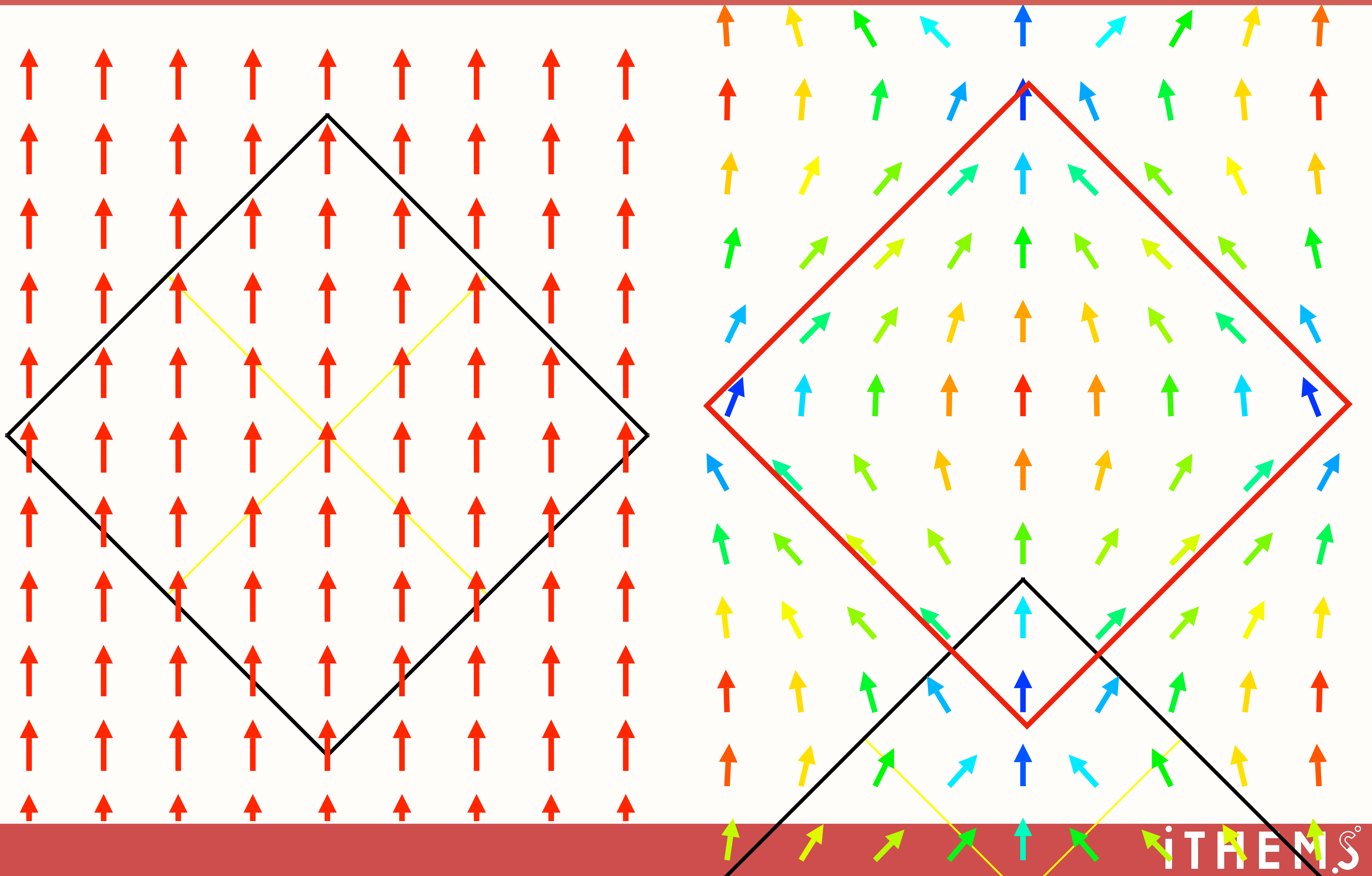
$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}$$

Minkowski CFT

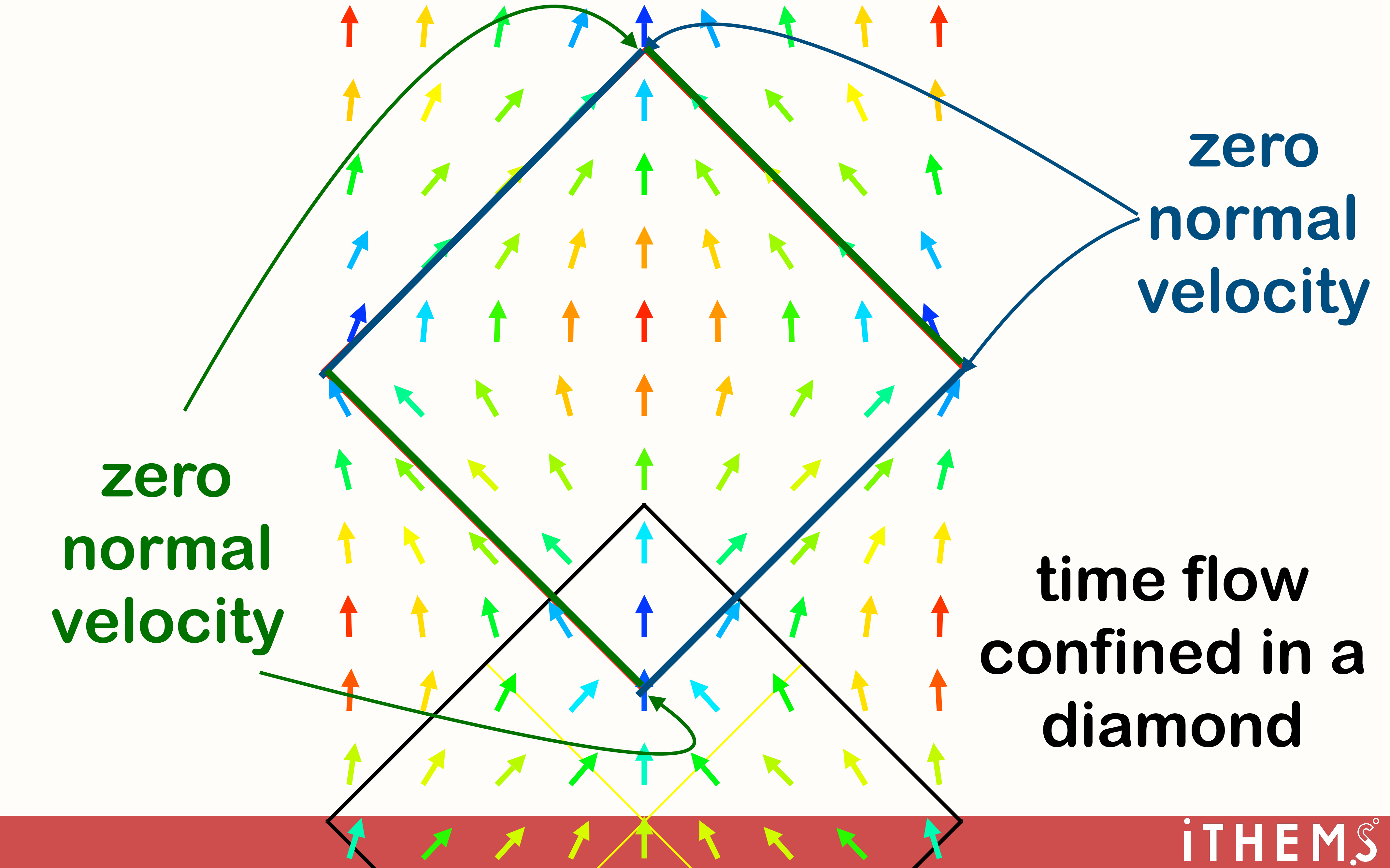


$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}$$

Minkowski CFT



Minkowski CFT



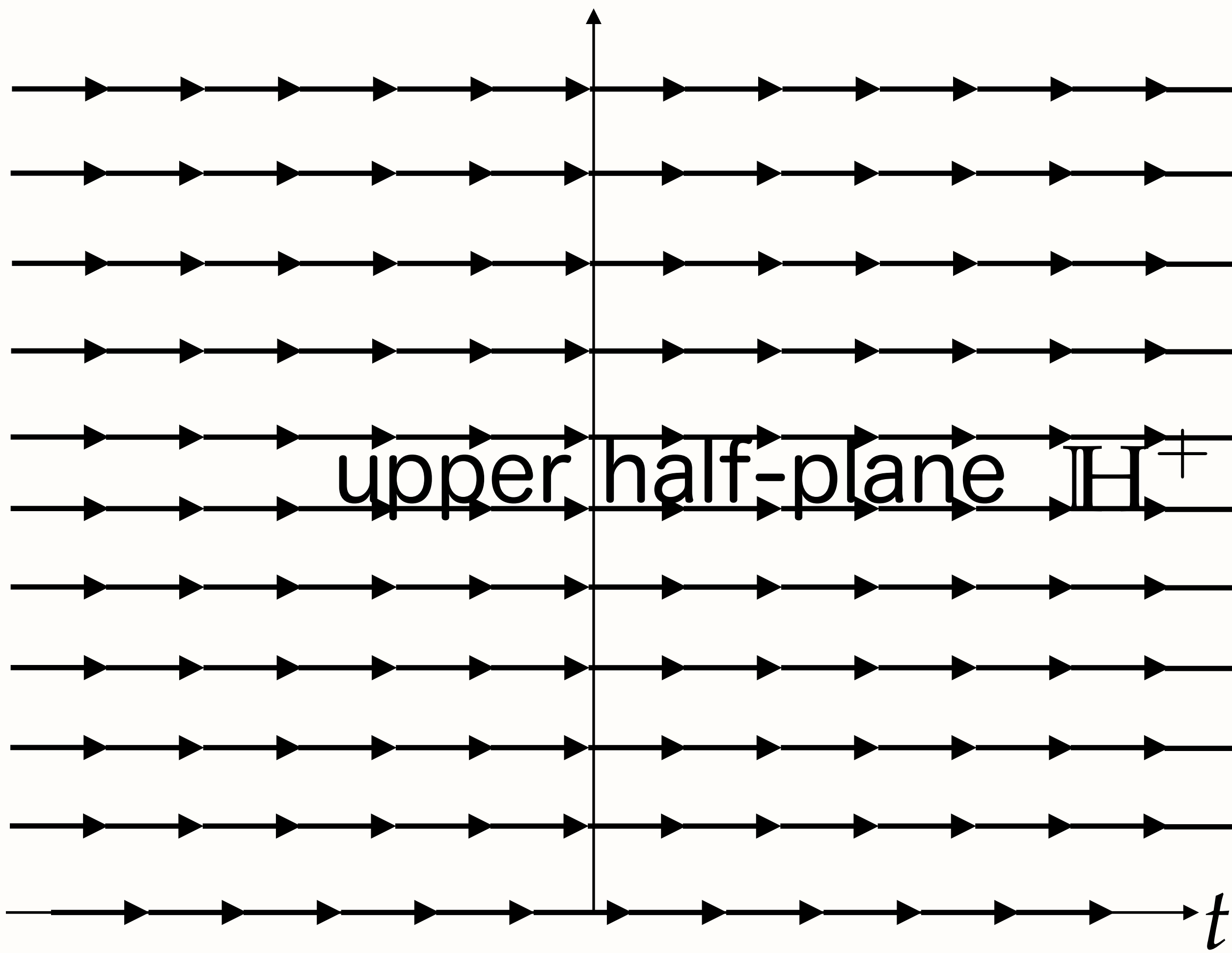
Outlook

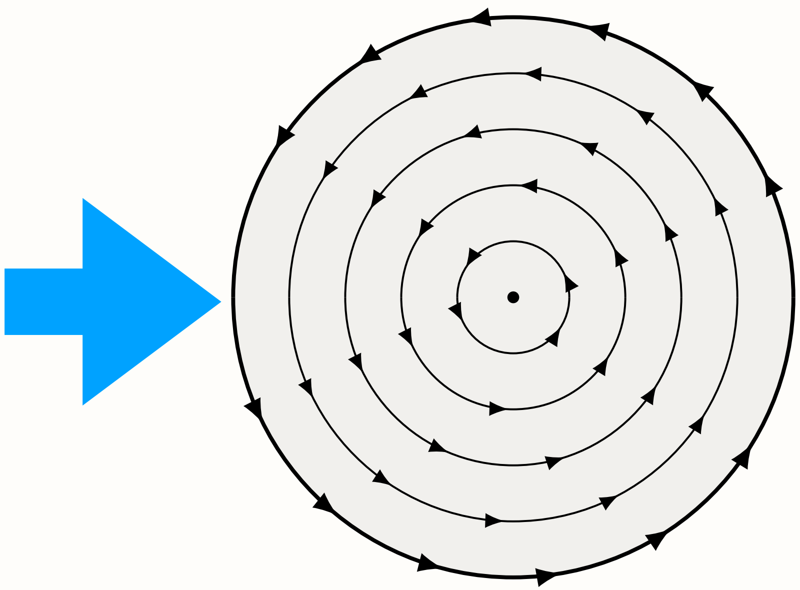
Time foliation over curved space

Two or more “time”s

Continuous Virasoro algebra

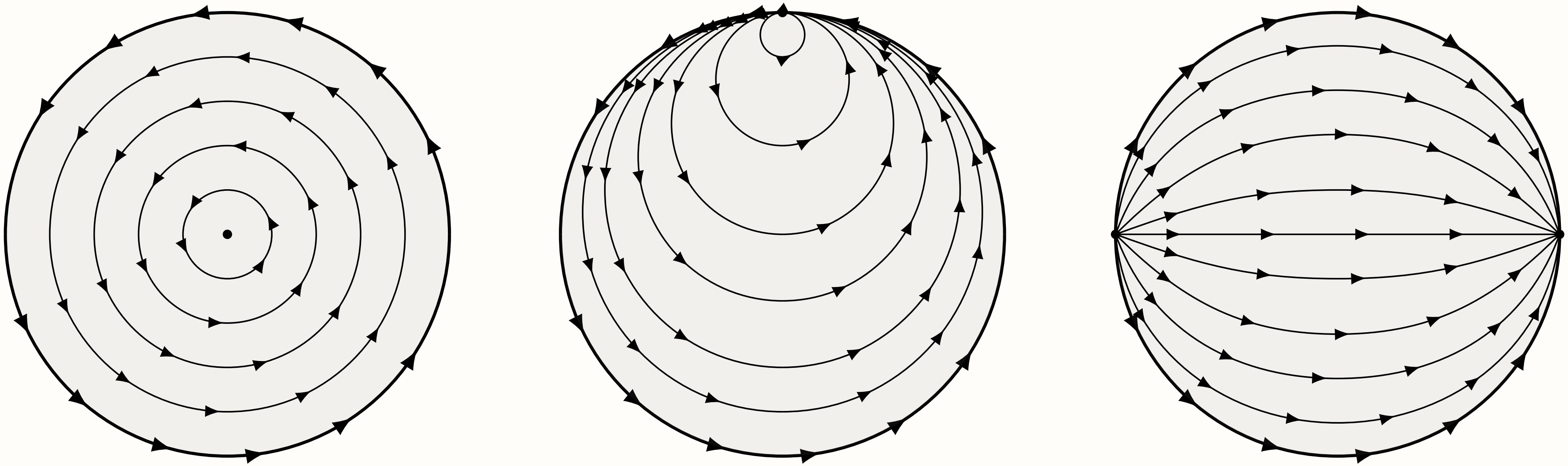
Hamiltonian Action



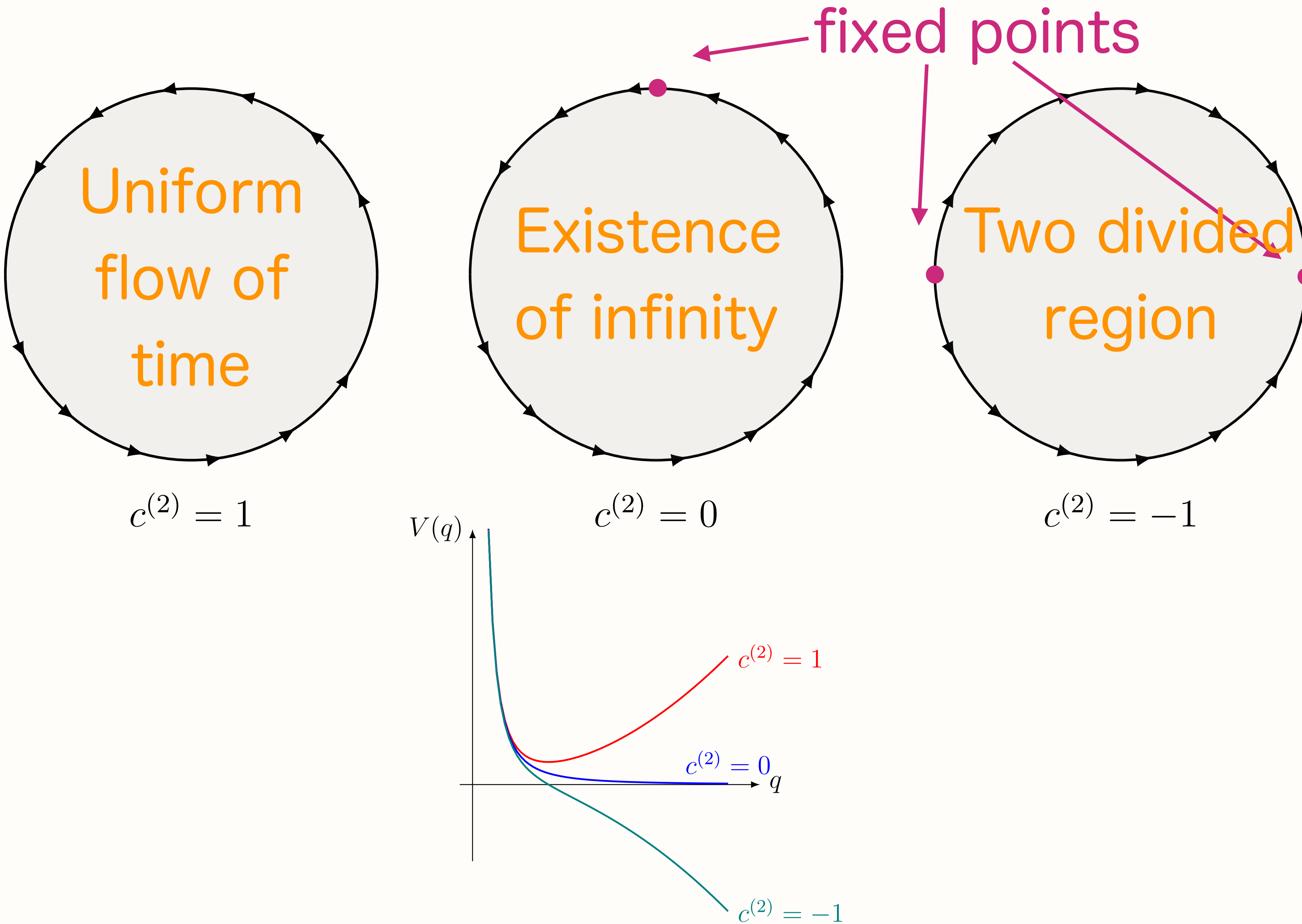


Map to
Poincare Disk

Automorphism on Poincare Disk



$c^{(2)} = 1$ $c^{(2)} = 0$ $c^{(2)} = -1$


fixed points
Uniform flow of time
Existence of infinity
Two divided region
$c^{(2)} = 1$
$c^{(2)} = 0$
$c^{(2)} = -1$
$V(q)$
q
$c^{(2)} = 1$
$c^{(2)} = 0$
$c^{(2)} = -1$