ディリクレ・ノイマン混合境界条件とhalf SSD

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Outline

Introduction

- What is SSD? What are special about SSD?
- Free-fermion models
- Results

Boundary conditions in lattice models

Half-SSD

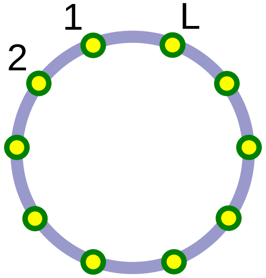
Summary

What is SSD (sine-square deformation)?

■ Setup and definitions

Consider a lattice model on a chain of length *L*. (PBC imposed)

- Uniform Hamiltonian $\mathcal{H}_0 = \sum_{j=1}^{L} h_j + \sum_{j=1}^{L} h_{j,j+1}$



- Chiral Hamiltonian

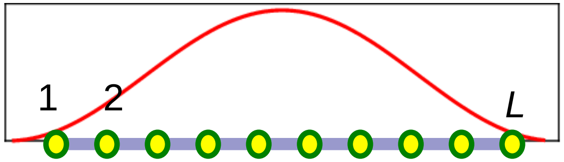
$$\mathcal{H}_\pm = \sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} h_j + \sum_{j=1}^{L} e^{\pm i\delta j} h_{j,j+1} \qquad \left(\delta = \frac{2\pi}{L}\right)$$

- SSD Hamiltonian

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

Gendiar *et al.*, *PTP* (2009-2010)
Hikihara, Nishino, *PRB* (2011)

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

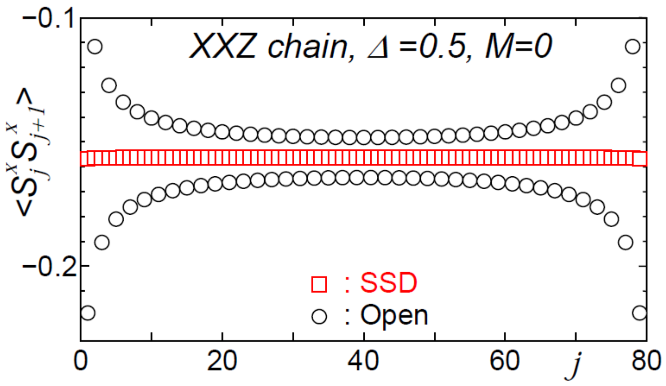


Ex.) Heisenberg chain

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\sum_{j=1}^{L}\left(1 - \frac{1}{2}e^{i\delta j} - \frac{1}{2}e^{-i\delta j}\right)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} = \sum_{j=1}^{L} \sin^2\left(\frac{\pi}{L}j\right)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

What are special about SSD?

- **Suppression of boundary effects**
  - Negligible Friedel oscillation
    Uniform g.s. correlations
  - Observed in ***1D critical systems***
    XXZ, Hubbard, Kondo-lattice, …
    Shibata, Hotta, *PRB* (2011)



Hikihara, Nishino, *PRB* **83**, 060414 (2011)

- **Scaling of entanglement entropy**

$$\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3} \ln \left[\frac{L}{\pi} \sin\left(\frac{\pi\ell}{L}\right)\right] + s_1$$

$$\mathcal{S}^{\mathrm{SSD}} \simeq \mathcal{S}^{\mathrm{PBC}}$$

- **Wavefunction overlap**

  Overlap between the g.s. of systems with PBC and SSD is almost 1.

$$\langle \Psi_{\mathrm{SSD}} | \Psi_{\mathrm{PBC}} \rangle \simeq 1$$

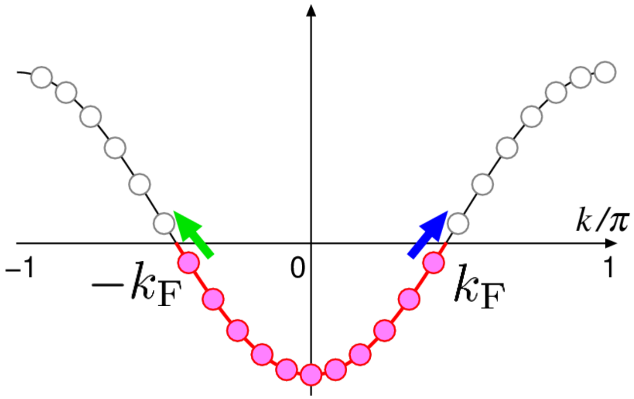
  - Rigorous proof for ***free-fermion models*** (XY, quantum Ising, …)
  - CFT interpretation: H.K., *JPA* **44**, 252001; **45**, 115003 (2011)

Free fermion chain with SSD (1)

- Uniform Hamiltonian

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} c_j^\dagger c_j$$

$c_j / c_j^\dagger$: annihilation/creation of fermion at *j*.



Fourier.tr.

$$\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k \qquad \epsilon(k) = -2t\cos k - \mu$$

G.S. of $\mathcal{H}_0$: Fermi sea ($\epsilon(k) < 0$ occupied) $\qquad \mathcal{H}_0|\mathrm{FS}\rangle = E_g|\mathrm{FS}\rangle$

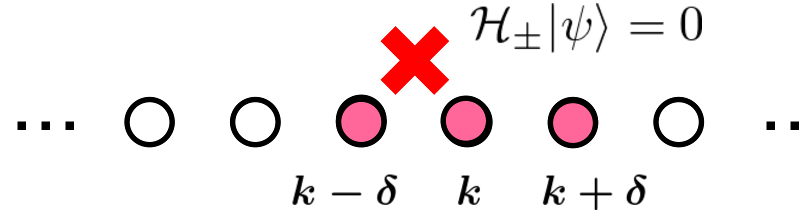
- Chiral Hamiltonian

$$\mathcal{H}_\pm = -t\sum_{j=1}^{L} e^{\pm i\delta j}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} c_j^\dagger c_j \quad \left(\delta = \frac{2\pi}{L}\right)$$

Momentum rep.

$$\mathcal{H}_\pm = e^{\mp i\delta/2}\sum_k \epsilon(k \mp \delta/2) c_k^\dagger c_{k\mp\delta}$$

$\mathcal{H}_\pm|\psi\rangle = 0$



If $\epsilon(k_{\mathrm{F}} + \delta/2) = \epsilon(-k_{\mathrm{F}} - \delta/2) = 0$, then $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0$. ($\because\ (c_k^\dagger)^2 = 0$)

Free fermion chain with SSD (2)

■ SSD Hamiltonian

$$\mathcal{H}_{\mathrm{SSD}} = -t \sum_{j=1}^{L-1} \sin^2\left(\frac{\pi}{L} j\right) (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu \sum_{j=1}^{L-1} \sin^2\left[\frac{\pi}{L}\left(j - \frac{1}{2}\right)\right] c_j^\dagger c_j$$

In terms of $\mathcal{H}_0$ & $\mathcal{H}_\pm$, $\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$

Fermi sea is annihilated by chiral Hamiltonians!

$$\boxed{\mathcal{H}_\pm |\mathrm{FS}\rangle = 0}$$

$$\mathcal{H}_{\mathrm{SSD}} |\mathrm{FS}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \cancel{\frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)}\right] |\mathrm{FS}\rangle = \frac{E_g}{2} |\mathrm{FS}\rangle$$

Fermi sea is an exact eigenstate of $\mathcal{H}_{\mathrm{SSD}}$!

■ Uniqueness of the ground state

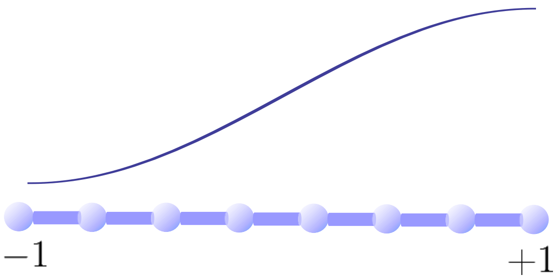
Fermi sea is ***the unique*** g.s. of $\mathcal{H}_{\mathrm{SSD}}$. $\mathcal{H}_0$ & $\mathcal{H}_{\mathrm{SSD}}$ share the same g.s.

Outline of proof)
Free-fermion chain → XY spin chain (via Jordan-Wigner)
Perron-Frobenius thm tells: (i) the ground state of $\mathcal{H}_{\mathrm{SSD}}$ is unique.
(ii) it has nonvanishing overlap with $|\mathrm{FS}\rangle$.

Results

- Any open/open correspondence?
- Yes, for free-fermion chain!
- But we need to add boundary potentials

$$H_0 = -\sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + \mathrm{h.c.}) - n_1 + n_L$$



$$H_{\mathrm{half}} = -\sum_{j=1}^{L-1} \sin^2\left(\frac{\pi j}{2L}\right) (c_j^\dagger c_{j+1} + \mathrm{h.c.}) + n_L$$

- H_0 and H_{half} share the same ground state

But why?

Outline

Introduction

Boundary conditions in lattice models

- Tight-binding model with boundary potential
- Analytically solvable cases, eigenfunctions

Half-SSD

Summary

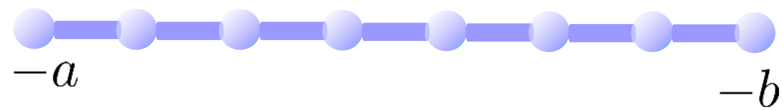
Tight-binding chain with boundary potential

■ Hamiltonian

$$H_0(a,b) = -\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1}) - a\,n_1 - b\,n_L$$

$c_j, c_j^\dagger$: spinless fermion ops.,

$n_j = c_j^\dagger c_j$: number op.



■ Tri-diagonal matrix

$$H_0(a,b) = -\boldsymbol{c}^\dagger\,\mathcal{T}(a,b)\,\boldsymbol{c}, \quad \boldsymbol{c}^\dagger = (c_1^\dagger, ..., c_L^\dagger)$$

- The `hopping' matrix *T* determines the 1-particle spectrum

$$\mathcal{T}(a,b) = \begin{pmatrix} a & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & 1 & b \end{pmatrix}$$

- Eigenvalue problem

$$\mathcal{T}(a,b)\boldsymbol{v} = \lambda\boldsymbol{v}$$

Analytically tractable?

Yes, for special (*a*, *b*)

List of exact solutions

- $a = b = 0$: Fixed-Fixed BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{L+1}\right), \quad m = 1, 2, ..., L$$

Martin-Delgado, Sierra, *Phys. Lett. B* **364**, 41 (1995)

- $a = b = 1$: Free-Free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{L}\right), \quad m = 0, 1, ..., L-1$$

- $a = 0$, $b = 1$: Fixed-Free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{2L+1}\right), \quad m = 1, 3, 5, ..., 2L-1$$

- $a = 1$, $b = -1$: Free-Anti-free BC

$$\lambda_m = 2\cos\left(\frac{m\pi}{2L}\right), \quad m = 1, 3, 5, ..., 2L-1$$

- $a = q$, $b = 1/q$: Saleur (proceedings, 1989)

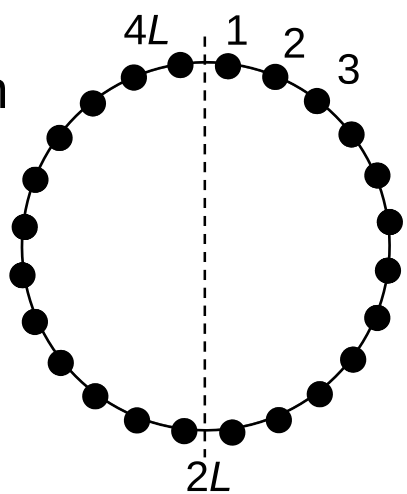
$$q + q^{-1}, \quad \lambda_m = 2\cos\left(\frac{m\pi}{L}\right) \quad m = 1, 2, ..., L-1$$

Appendix in HK, Schuricht, Takahashi, *PRB* **92**, 115137 (2015)

How to get eigenfunctions

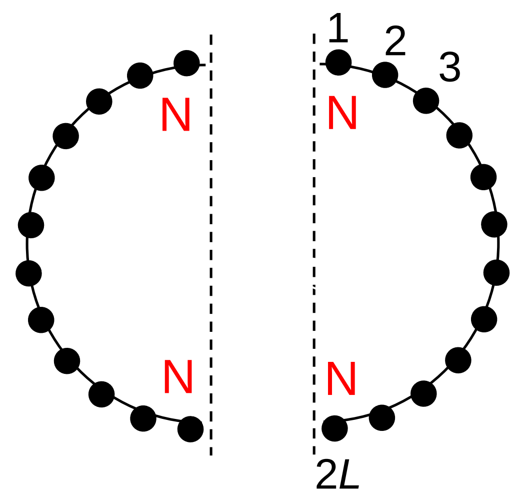
- Orbifolding?

Tight-binding on a periodic chain of length $4L$

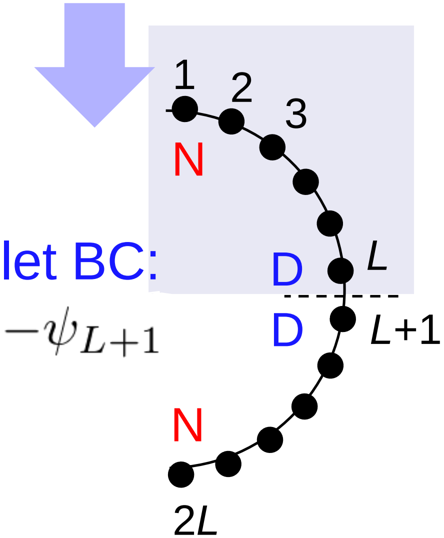


Neumann BC.:

$$\psi_1 = \psi_{4L}, \quad \psi_{2L} = \psi_{2L+1}$$



One can get eigenfunctions of $T(1,-1)$ from the plane wave solutions on a periodic ring!



Dirichlet BC:

$$\psi_L = -\psi_{L+1}$$

- Eigenfunctions

$$\psi_j^{(k)} = \sqrt{\frac{2}{L}} \cos\left[\frac{\pi(2j-1)(2k-1)}{4L}\right]$$

$$1 \le j \le L, \quad 1 \le k \le L$$

Outline

Introduction

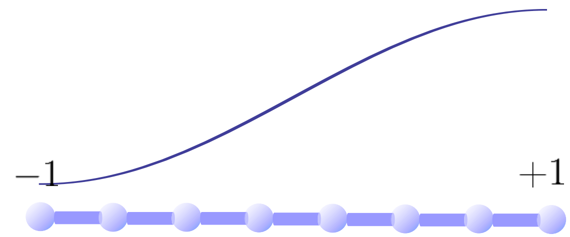
Boundary conditions in lattice models

Half-SSD

- What is half-SSD?
- Open/Half-SSD correspondence
- Self-duality, commuting property

Summary

What is half SSD?



Setup

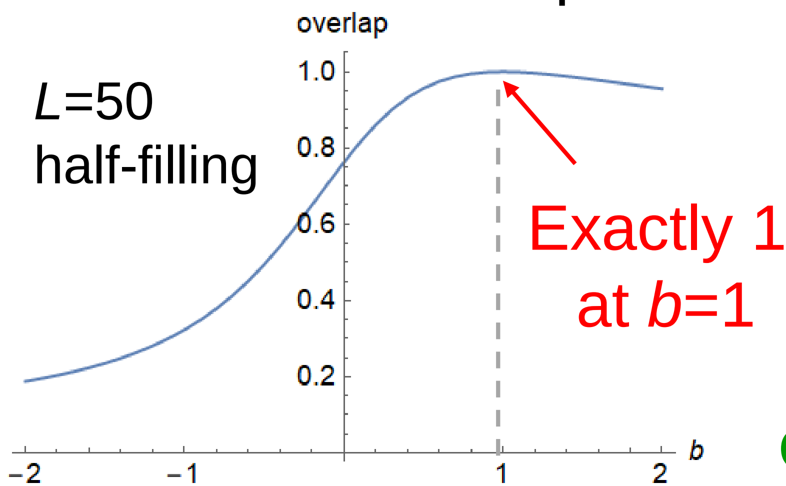
- Open-chain Hamiltonian

$$H_0(1,-1) = -\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + \mathrm{h.c.}) - n_1 + n_L$$

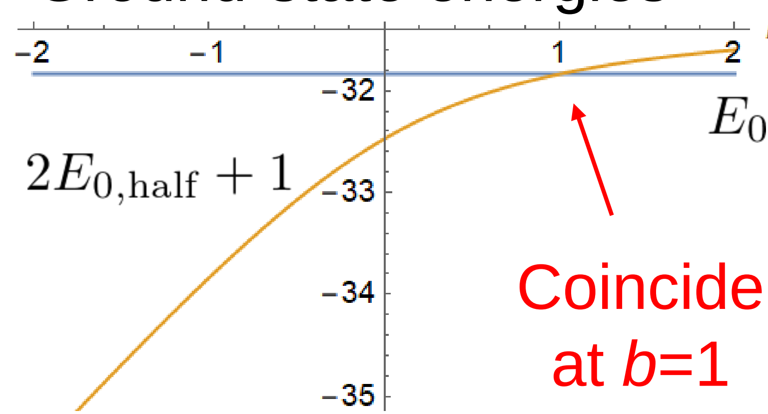
- Half-SSD Hamiltonian with boundary potetial

$$H_{\mathrm{half}}(b) = -\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi j}{2L}\right)(c_j^\dagger c_{j+1} + \mathrm{h.c.}) + b\,n_L$$

Ground-state overlap



Ground-state energies



Open/half-SSD correspondence!

What happens at *b*=1?

- $H_{\rm half}$ in the H_0-diagonal basis (L: even)

$$H_{\rm half}(b) = \boldsymbol{c}^\dagger \, \mathcal{H}_{\rm half}(b) \, \boldsymbol{c},$$

$$\mathsf{H}_{k,l} = \langle \psi^{(k)}, \mathcal{H}_{\rm half}(1) \psi^{(l)} \rangle$$

Eigenfunctions of T(1,-1)

$$\psi_j^{(k)} = \sqrt{\frac{2}{L}} \cos\left[\frac{\pi(2j-1)(2k-1)}{4L}\right]$$

- Diagonal matrix elements

$$\mathsf{H}_{k,k} = -\cos\left[\frac{\pi(2k-1)}{2L}\right] + \frac{1}{2}(\delta_{k,1}+\delta_{k,L}), \quad (k=1,\ldots,L)$$

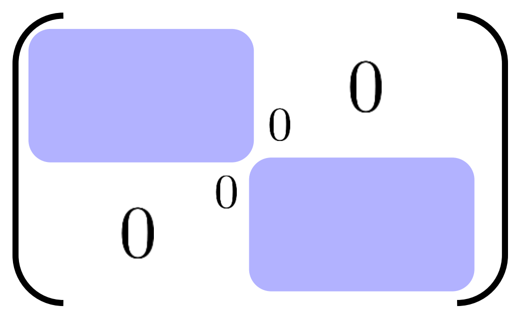
- Off-diagonal elements

$$\mathsf{H}_{k,k+1} = \mathsf{H}_{k+1,k} = \frac{1}{2}\cos\left(\frac{\pi k}{L}\right), \quad (k=1,\ldots,L-1)$$

The other elements are all zero.

Clearly

$$\mathsf{H}_{L/2,L/2+1} = \mathsf{H}_{L/2+1,L/2} = 0.$$



Fermi sea of H_0 is the g.s. of $H_{\rm half}$!

Self-dual Hamiltonian

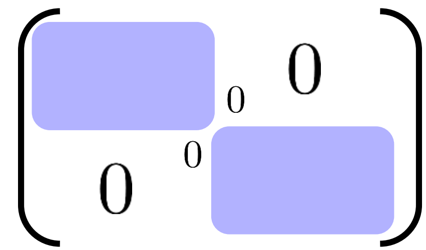
■ Interpolation between $H_{\rm half}$ and H_0

$$H_{\rm half}(1) = \frac{1}{2}H_0(1,-1) + \frac{1}{2}\underbrace{\left[\sum_{j=1}^{L-1}\cos\left(\frac{\pi j}{L}\right)(c_j^\dagger c_{j+1} + \text{h.c.}) + n_1 + n_L\right]}_{H_1}$$

- 1-parameter Hamiltonian

$$H(\alpha) = \frac{1}{2}(H_0(1,-1) + \alpha H_1)$$

N.B.: $H(0) = \frac{1}{2}H_0(1,-1),\ \ H(1) = H_{\rm half}(1)$

$H_1 =$ 

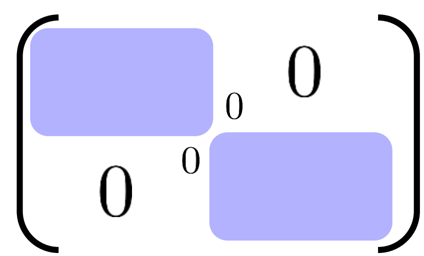
■ Self-duality of H_1

- Eigen-operators of H_0

$$\tilde{c}_k = \sum_{j=1}^{L}\psi_j^{(k)}c_j, \quad \tilde{c}_k^\dagger = \sum_{j=1}^{L}\psi_j^{(k)}c_j^\dagger$$

- Eigen-space Hamiltonian (H_1)

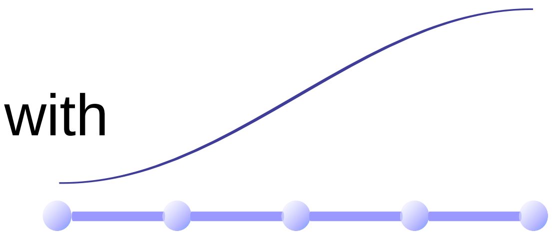
$$H_1 = \sum_{j=1}^{L-1}\cos\left(\frac{\pi j}{L}\right)(\tilde{c}_j^\dagger\tilde{c}_{j+1} + \text{h.c.}) + \tilde{c}_1^\dagger\tilde{c}_1 + \tilde{c}_L^\dagger\tilde{c}_L$$



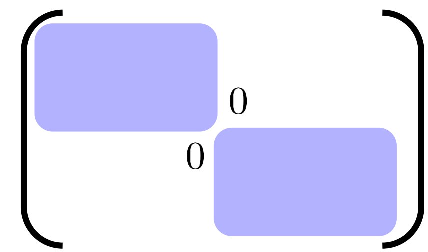
Takes the same form as H_1 in real-space!

Summary

- Studied tight-binding chain with half-SSD
- The models shares the same ground state with the tight-binding chain with special b.c.



- The b.c. = mixed Dirichlet-Neumann b.c.
- Decoupling structure in the `eigen-space'



Future directions

- Extension to finite chemical potential
- Field theory: bosonization, CFT, …
- Algebra? Anything to do with modular *S*-matrix? $\psi_j^{(k)} = S_j^k \ (?)$
- Extension to other boundary conditions?
  Other mixed b.c. What about Robi b.c.? $(\alpha\psi + \beta\psi')\big|_{\partial\Omega} = 0$
- What are they good for?