Quantum-Classical correspondence and Energy Scale Deformations

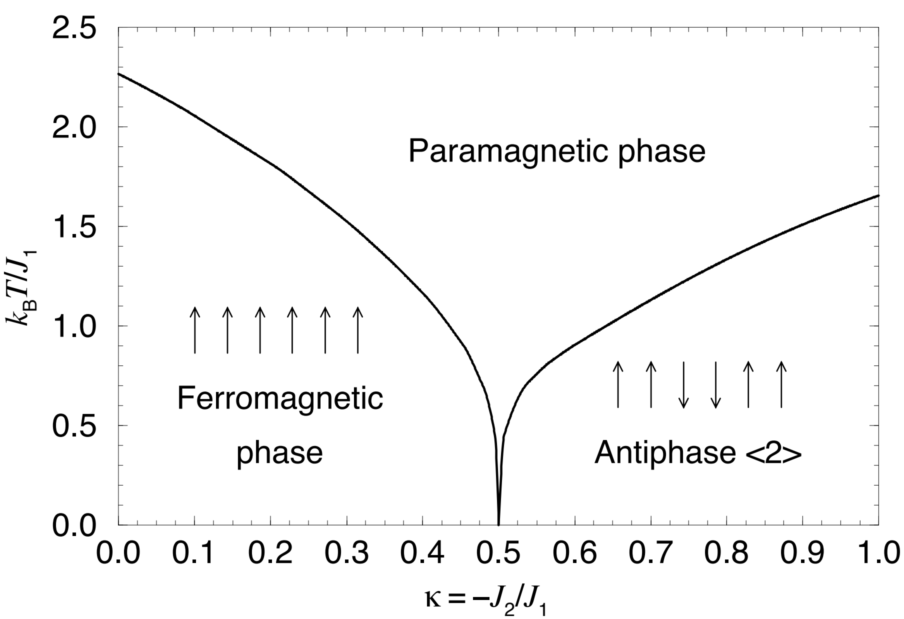
Tomotoshi Nishino (Kobe Univ.)
Roman Krcmar (SAS)
Andrej Gendiar (SAS)
… anonymous referee …

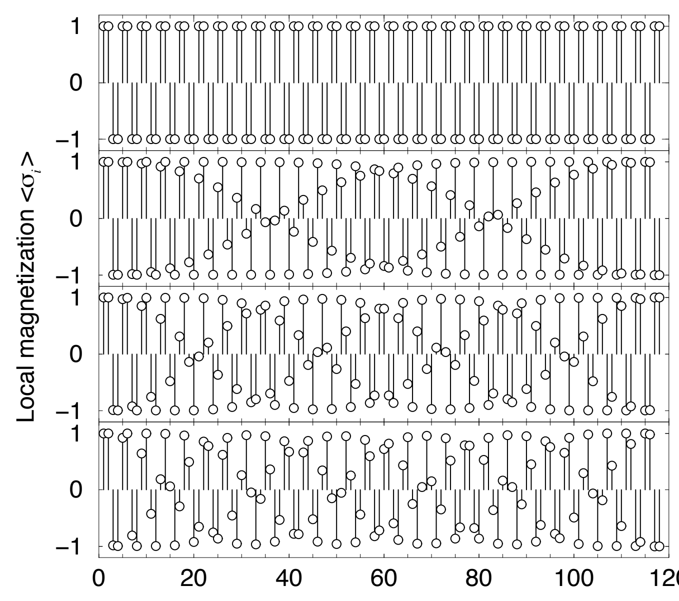
arXiv:0810.0622

* Uniform Hamiltonian does not always have uniform ground state.

- **Charge/Spin density wave, commensurate or incommensurate**
- **ex. Axial Next Nearest Neighbor Ising (ANNNI) model**

$$\mathcal{H} = -J_1 \sum_{i,j} \sigma_{i,j}(\sigma_{i+1,j} + \sigma_{i,j+1}) - J_2 \sum_{i,j} \sigma_{i,j}\sigma_{i+2,j}$$





Energy Scale Deformation

* There is a modulated Hamiltonian whose ground state is uniform.

- empty state of any Fermionic system (too trivial!)

- (modulated/inhomogeneous) AKLT Hamiltonian

since H = sum of projectors, and pre factor can be arbitral

- Slow energy scale modulation would not affect a gapped ground state

if the modulation is slow enough (or gap is wide enough)

- Exponential Deformation (Wilson, …, Okunishi)

wilson lattice arXiv:1001.2594

$$\mathcal{H}_\lambda = \sum_{n=1}^{N-1} e^{\lambda n} (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1})$$

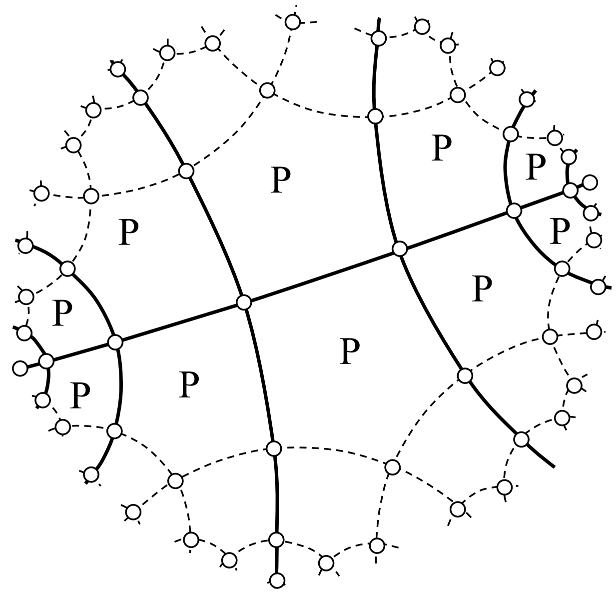
general framework arXiv:cond-mat/0702581

$$H_N(\Lambda) = \sum_{n=1}^{N-1} \Lambda^{N-n-1} h_{n,n+1},$$

a classical counterpart: Hyperbolic Lattice

arXiv:0704.1949

Ising model on Hyperbolic Lattice



- there is ferro-para phase transition
- always off critical
- row-to-row transfer matrix can be defined

- is it possible to find out the corresponding quantum Hamiltonian? (I have no answer)

probably, in anisotropic limit (how to define this limit?), one reaches the hyperbolic deformation. arXiv:0808.3858

$$H^{\cosh}(\lambda) = \frac{1}{2}\left[H^{\exp}(\lambda) + H^{\exp}(-\lambda)\right]$$

$$= \sum_{j=-N}^{N} \cosh j\lambda \; h_{j,j+1} \, .$$

ground-state is uniform, except for the edge state, as it was observed in the case of exp. deformation.

a path to "spherical" deformation

* Corner Hamiltonian ~ Entanglement Hamiltonian

- Okunishi proposed a quantum counterpart of CTMRG

$$K_{\rm N} = \sum_{n=1}^{N-1} n h_{n,n+1}, \quad \text{cond-mat/0507195}$$

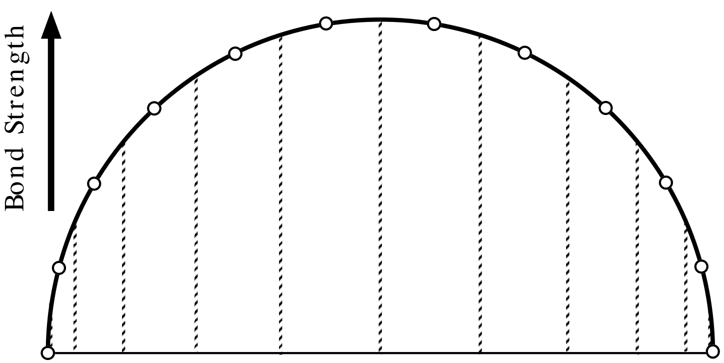
- Hyperbolic "deformation" can be considered

$$H^{\sinh}(\lambda) = \sum_{j=-N}^{N} \sinh j\lambda \; h_{j,j+1} \,, \quad \text{arXiv:0808.3858}$$

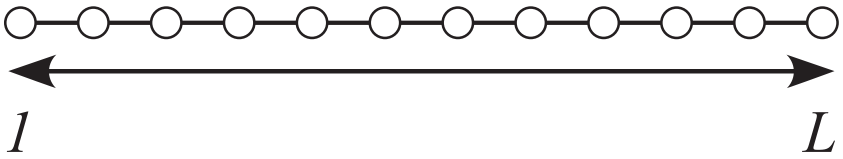
* History in physics suggests the generalization to trigonometric deformations

$$H_{\rm Sph.} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell) \, h_{\ell,\ell+1} \quad \text{arXiv:0810.0622}$$

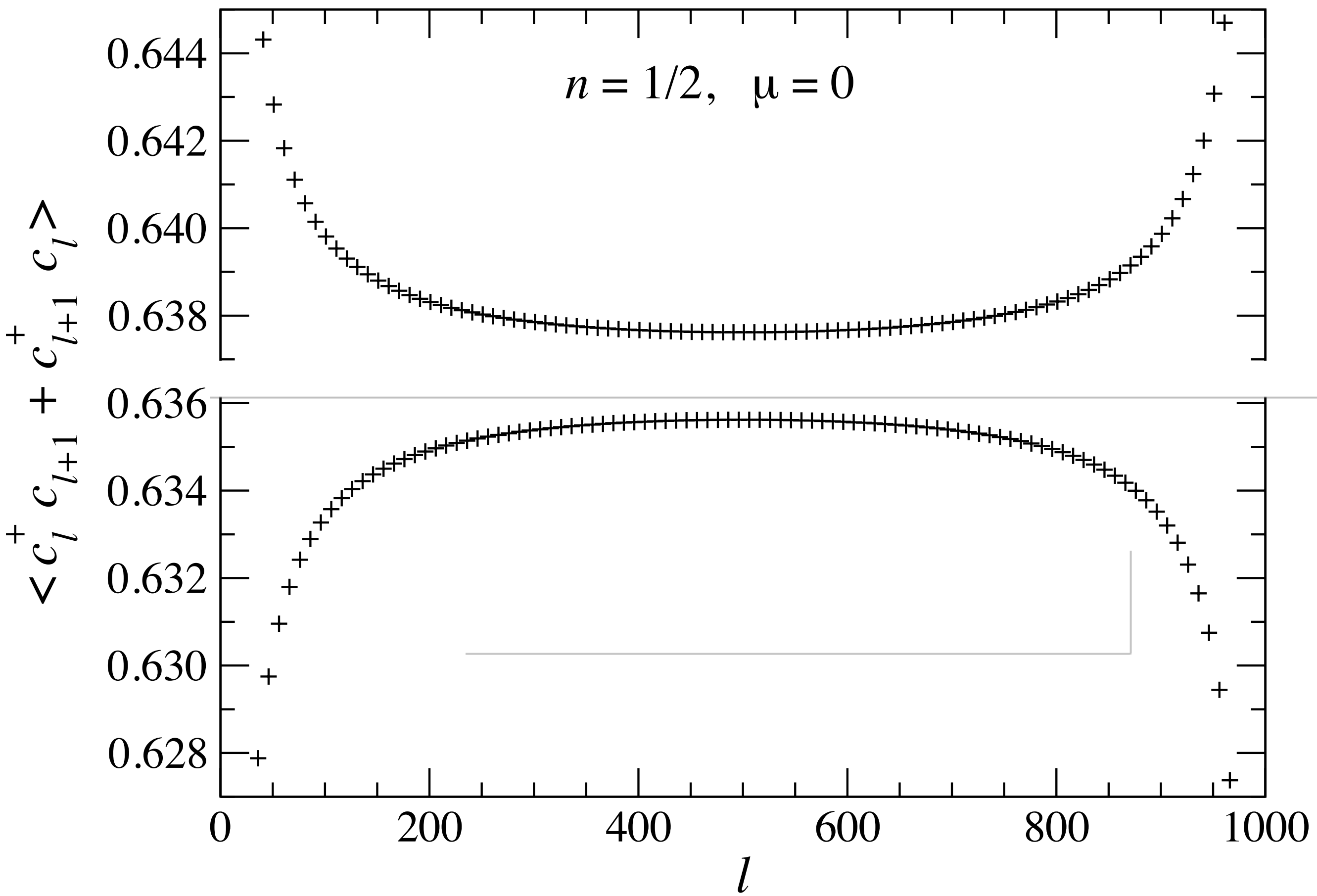
… well, the prototype was "cosine deformation", and not squared.
How can one use the deformation? (I don't know.)



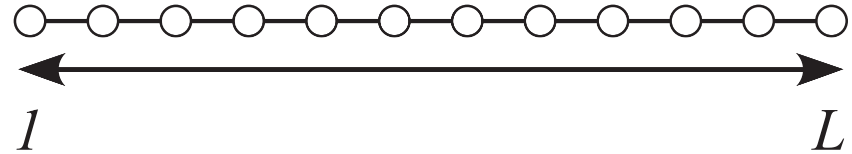
arXiv:0810.0622



最近接格子点間の「相関関数」を求めてみる。N=1000 サイトの系での計算結果は？「境界効果」であるフリーデル振動が、内部まで浸透していることがわかる。(金属表面で電子密度が振動するのも同じようなもの)



Smooth Boundary Condition



飛び移り振幅 -t を、系の両端で小さくすれば、上手く「ターミネート」できるのではないか？

PHYSICAL REVIEW LETTERS

VOLUME 71 | 27 DECEMBER 1993 | NUMBER 26

Smooth Boundary Conditions for Quantum Lattice Systems

M. Vekić and S. R. White

Department of Physics, University of California, Irvine, California 92717

(Received 1 September 1993)

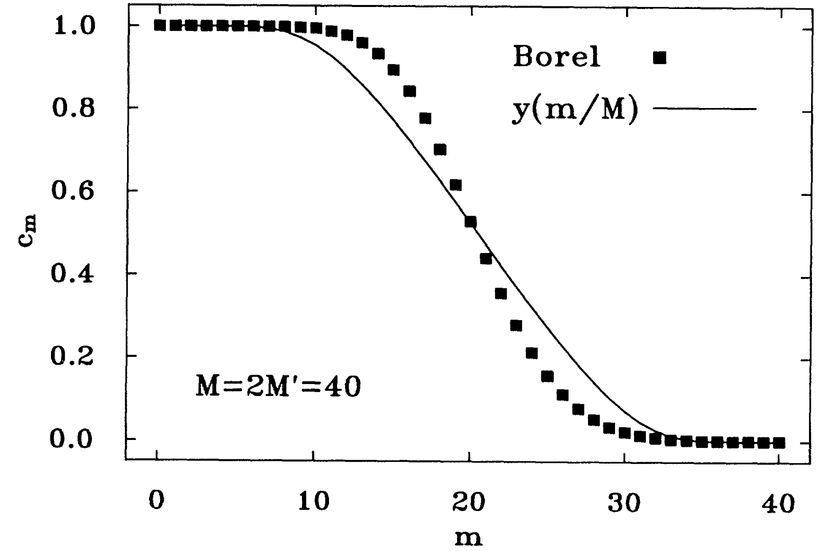
We introduce a new type of boundary conditions, *smooth boundary conditions*, for numerical studies of quantum lattice systems. In a number of circumstances, these boundary conditions have substantially smaller finite-size effects than periodic or open boundary conditions. They can be applied to nearly any short-ranged Hamiltonian system in any dimensionality and within almost any type of numerical approach.

PACS numbers: 02.70.−c, 05.30.Fk, 75.10.Jm

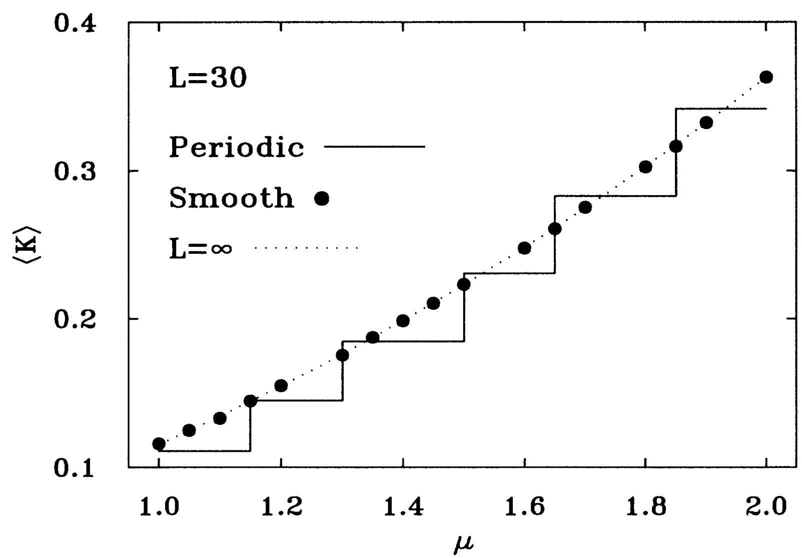
まあまあ上手く行く

White の成果

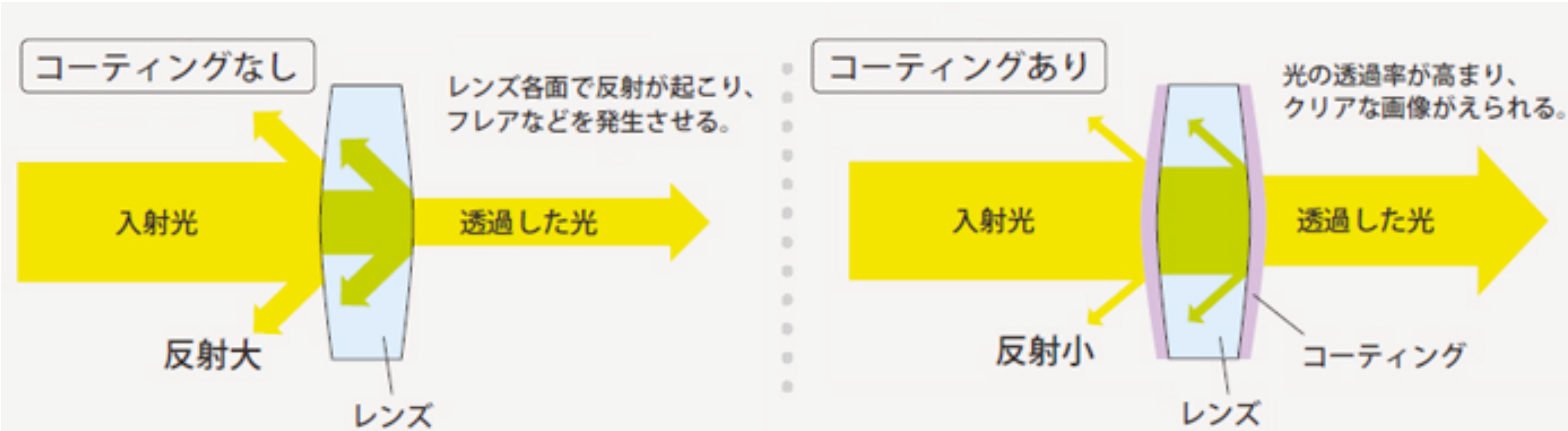
飛び移り振幅 -t の、系の両端でのスムージング関数



化学ポテンシャル変化に対する粒子密度の変化



レンズのコーティングもまた同じ



← 望遠鏡の善し悪しは、対物レンズのコーティングを見ると、おおよそ推測できることが多い。
（粗悪品は値段の割に口径が大！）

a path to "spherical" deformation

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- Okunishi proposed a quantum counterpart of CTMRG

$$K_{\mathrm{N}} = \sum_{n=1}^{N-1} n h_{n,n+1},$$

cond-mat/0507195

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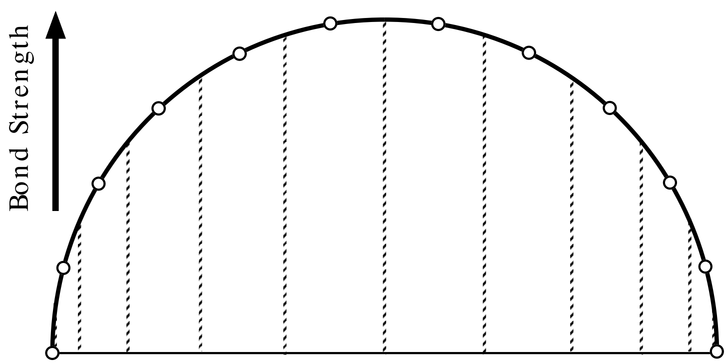


arXiv:0810.0622

Spherical Deformation for One-dimensional Quant

Andrej Gendiar, Roman Krcmar, Tomotoshi Nishino

(Submitted on 3 Oct 2008 (v1), last revised 27 Dec 2010 (this version, v6))

Submission history

From: Andrej Gendiar [view email]
[v1] Fri, 3 Oct 2008 12:09:55 UTC (58 KB)
[v2] Mon, 30 Mar 2009 14:55:38 UTC (71 KB)
[v3] Fri, 19 Jun 2009 14:47:53 UTC (308 KB)
[v4] Tue, 14 Jul 2009 17:15:13 UTC (326 KB)
[v5] Thu, 16 Jul 2009 16:57:13 UTC (326 KB)
[v6] Mon, 27 Dec 2010 08:05:20 UTC (447 KB)

[v1] $$H_{\rm S}^{N} = -t \sum_{\ell=-N/2}^{N/2-2} \cos\left(\frac{\ell+1}{N-1}\pi\right)\left(c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell\right)$$

… finally we reach sin^2 form, … almost ACCIDENTALLY

Errara published in Prog. Theor. Phys. **123** (2010), 393.

Errata

Spherical Deformation for one-dimensional Quantum Systems

Andrej Gendiar, Roman Krcmar, and Tomotoshi Nishino

Prog. Theor. Phys. **122** (2009), 953.

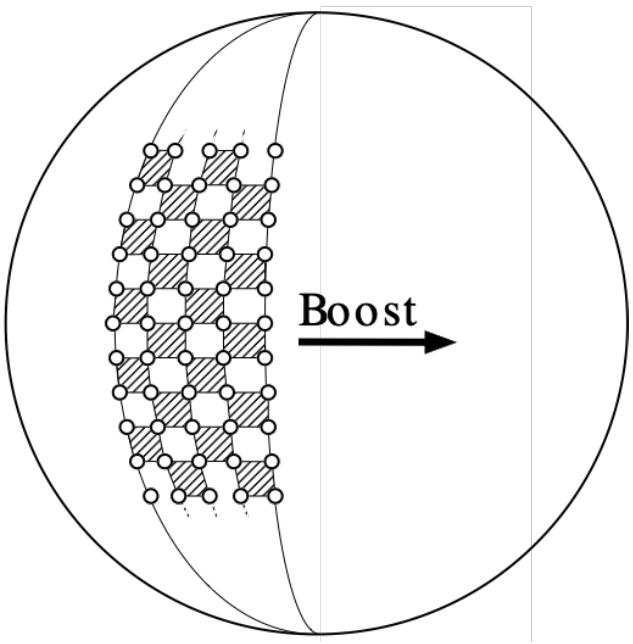
In the article we have published, we studied the finite-size correction to the energy per site E^N/N for the spherically deformed free fermion lattice, whose Hamiltonian is given by

$$\hat{H}_{\rm S}^{(n)} = \sum_{\ell=1}^{N-1}\left[\sin\frac{\ell\pi}{N}\right]^n\left(-t\,\hat{c}_\ell^\dagger\hat{c}_{\ell+1} - t\,\hat{c}_{\ell+1}^\dagger\hat{c}_\ell - \mu\frac{\hat{c}_\ell^\dagger\hat{c}_\ell + \hat{c}_{\ell+1}^\dagger\hat{c}_{\ell+1}}{2}\right) \quad (1)$$

What happened?

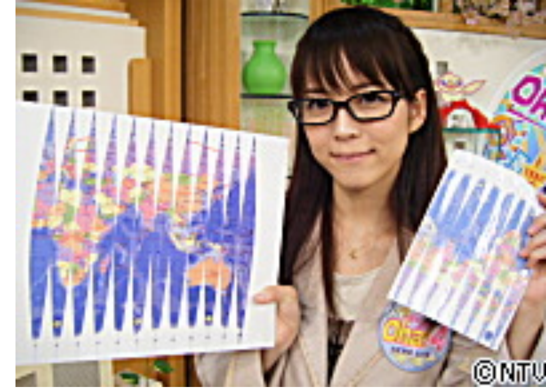
- I visited Aachen, to discuss with Andrej Gendiar in 2008.

… we considered a way of reducing the boundary effect in 1D chain.



The following picture came up, though I do not understand what it is even now. (open problem)

a sphere has no border



let us focus on the width of each piece of paper.

$$\mathcal{H}^{(N)}_{\rm sine} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N}\right)\right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Major contribution came from Andrej Gendiar

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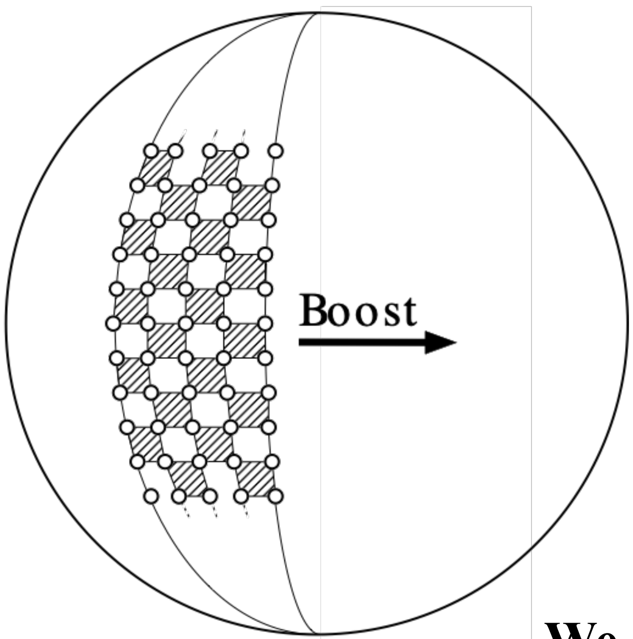
… we considered a way of reducing the boundary effect in 1D chain.



The following picture came up, though I do not understand what it is even now. (open problem)

any way, we checked the "cosine deformation" on the free fermion lattice, and confirmed that it reduces the boundary effect.

$$H_{\mathrm{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell)\, h_{\ell,\ell+1}$$

We report the result as [v1] of arXiv:0810.0622

ATTENTION: we submit [v1] to Prog. Theor. Phys.
Referee pointed that the boundary effect is reduced, but still there is.

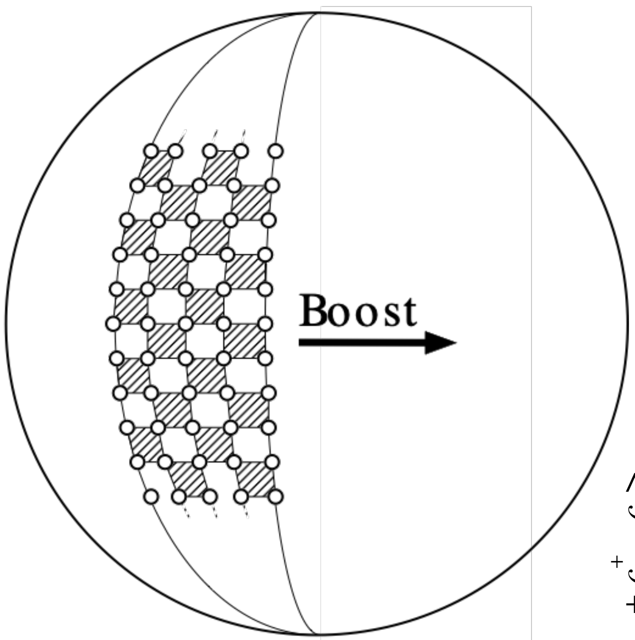
- Andrej proposed to consider cos^n also, since the function falls to 0 MORE SMOOTHLY than cos^1.

- I denied Andrej's proposal, since cos^n contradict the above SPHERE.

What happened?

- I visited Aachen, to discuss with Andrej Gendiar in 2008.

… we considered a way of reducing the boundary effect in 1D chain.



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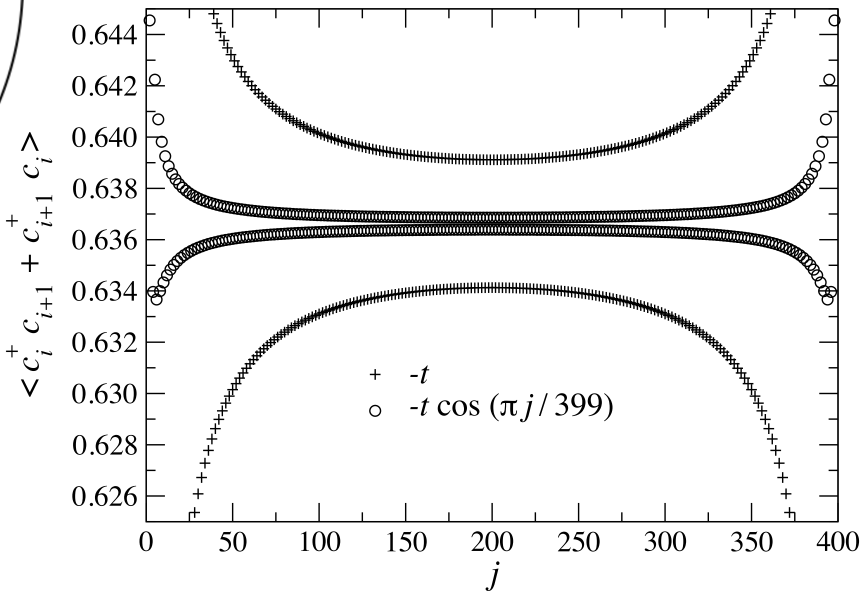


Figure 2. Expectation value $\langle c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell \rangle$ of the spherically deformed lattice Fermion model when $N = 400$. For comparison, we also plot the same expectation value for the undeformed case.

What happened? - Andrej was right, and there is one another side story.

[ERRATA] arXiv:0810.0622

In the article we have published, we studied the finite-size correction to the energy per site E^N/N for the spherically deformed free fermion lattice, whose Hamiltonian is given by

$$\hat{H}_{\rm S}^{(n)} = \sum_{\ell=1}^{N-1} \left[\sin \frac{\ell\pi}{N} \right]^n \left(-t\, \hat{c}_\ell^\dagger \hat{c}_{\ell+1} - t\, \hat{c}_{\ell+1}^\dagger \hat{c}_\ell - \mu \frac{\hat{c}_\ell^\dagger \hat{c}_\ell + \hat{c}_{\ell+1}^\dagger \hat{c}_{\ell+1}}{2} \right) \tag{1}$$

for the case $n = 1$. While we proceeded to a further study on the spherical deformation, we noticed the data shown in Figs. 2-7 were incorrect, and these figures corresponded to the Hamiltonian for the case $n = 2$. This error happened due to a very primitive confusion in the file name of computational source codes, and we misused the data with $n = 2$, instead of $n = 1$. We show appropriate data for the typical case $\mu = 0$, which corresponds to the half filling.

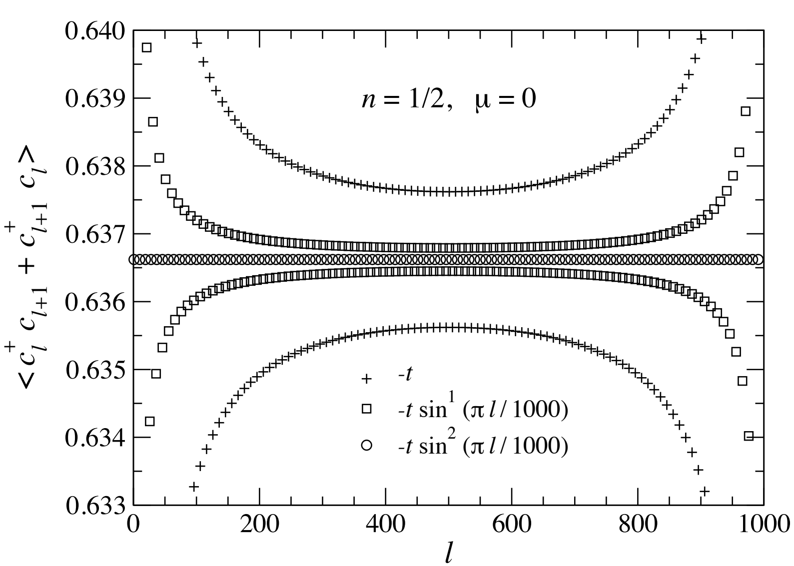


Fig. 1. Bond correlations at half filling calculated for $\hat{H}_{\rm S}^{(n)}$ with $n = 0$, 1, and 2.

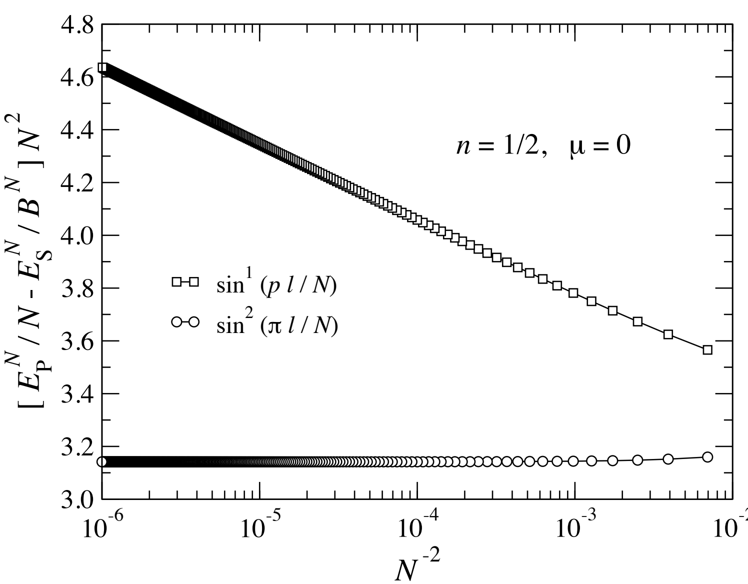


Fig. 2. Finite-size corrections to the energy.

Home Works (Conjectures)

Extension to higher dimensional system

- It is always possible to consider Hyperbolic lattice or deformation.

- Slowest modulation on N-dimensional sphere would be an extension of SSD.

Trotter decomposition

- What is the right Trotter decomposition between curved surface with constant curvature and corresponding quantum (lattice) system.

Fuzzy space

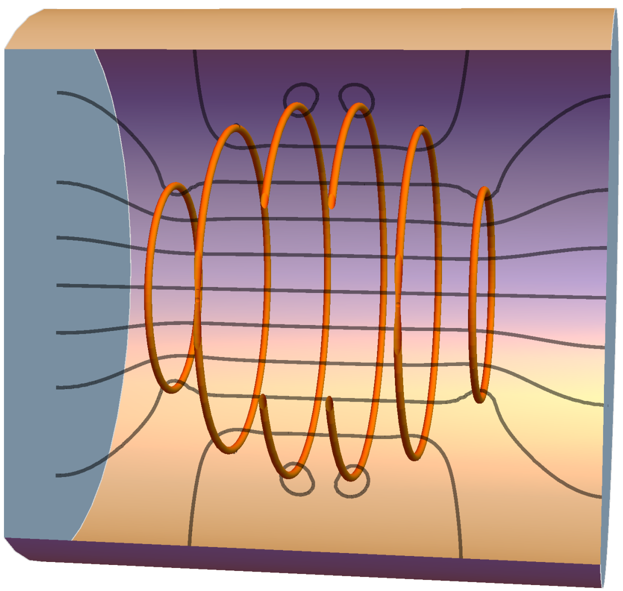
- How does non commutable space can be deformed in the manner of SSD?

[The world of Classical Physics is quite Wide]

electric magnet: should it be a cylinder?

What is the most appropriate form for the high field magnet?

Liu et al. arXiv:1907.03539



Spherical coil?

Hyperbolic helical coil?

Do find something rectangular/cylindrical

fill this space.

try to find on SNS.

Do find something rectangular/cylindrical

You are looking at rectangular screen.

u phone, also.

以下、付録

境界条件 (Boundary Condition) というもの

同じ水面でも、その性質は容れ物に
よってエラく変化する。

a glass of water

a `pacific’ of water

drawing by active boundary

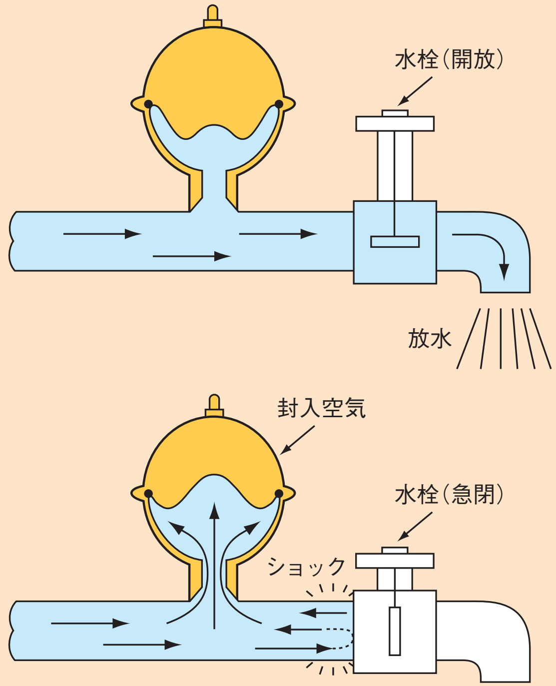
注) 文字が現れるのは一瞬だけ→

境界の効果は「反射」として現れる

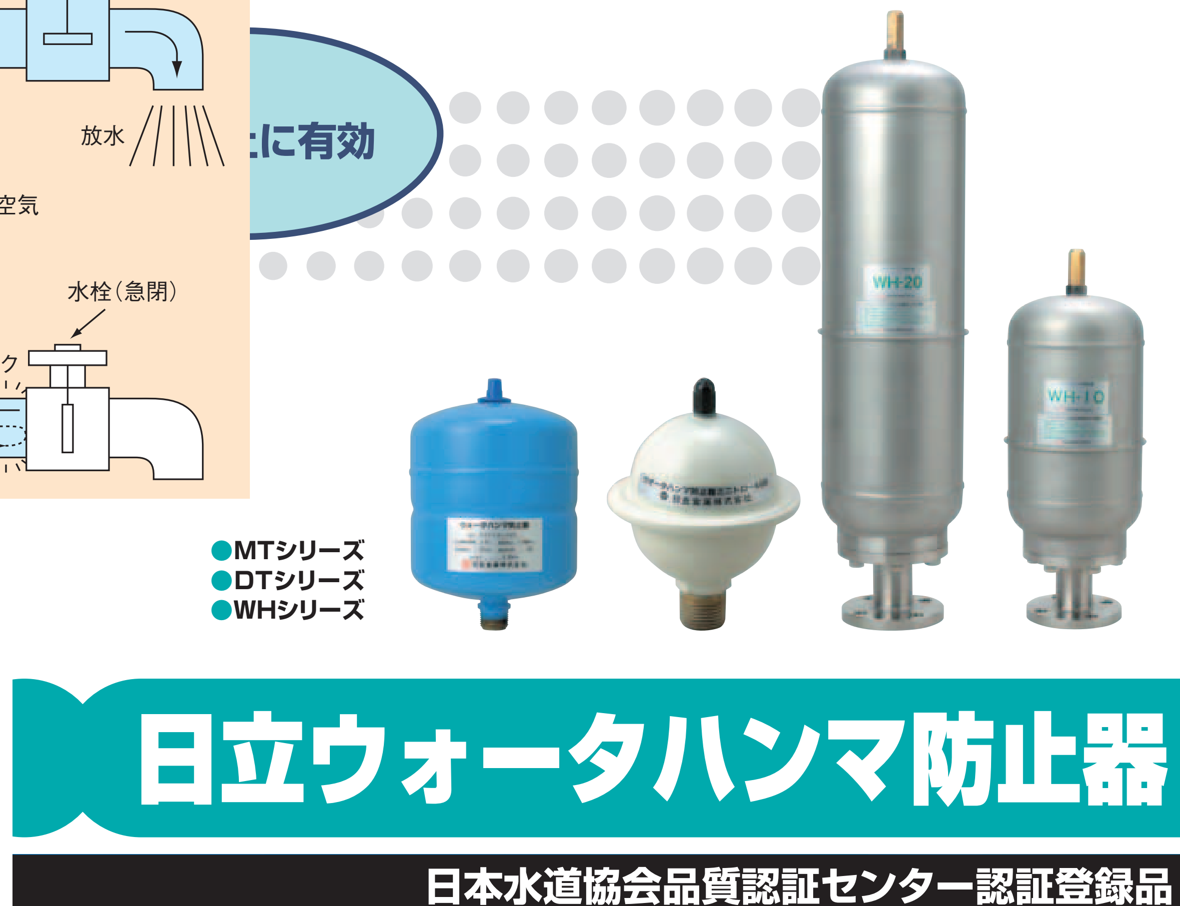
→ 反射を減じて「無限を演出」したくなることもある

水栓が開かれて、水が流れている状態。

水栓が急閉止されると、水がミニトロールの中に流入し、ダイアフラムを介して封入空気が圧縮され水撃が防止・軽減されます。



例えば水道では（通称）蛇口という境界があって、急に閉じると強い圧力波を発生させてしまう。

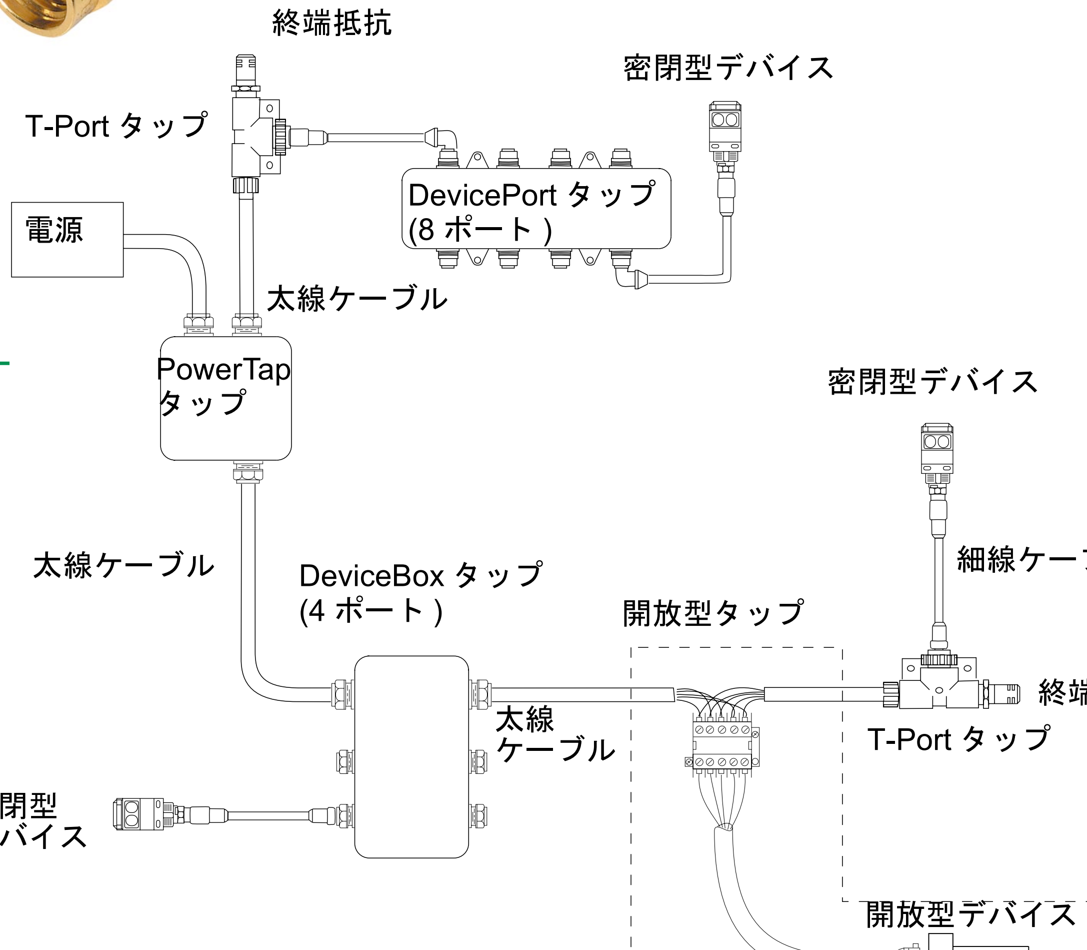


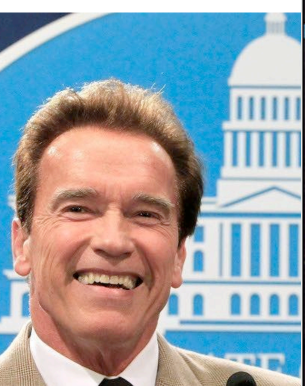
そこで、こんなものが裏で使われている、ことがある。

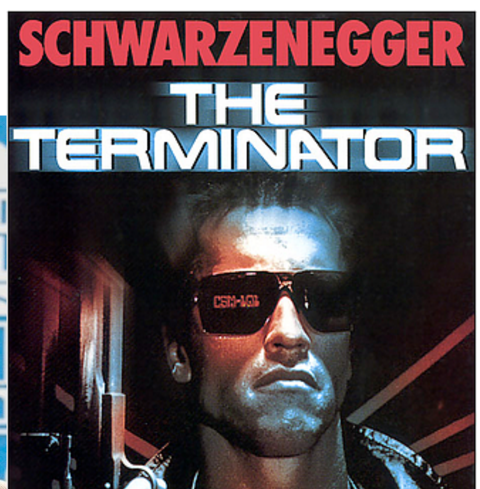
電気回路（や音響回路など）のインピーダンス整合も
境界からの信号反射を減じるための工夫である。

昔なつかしい SCSI のターミネーター

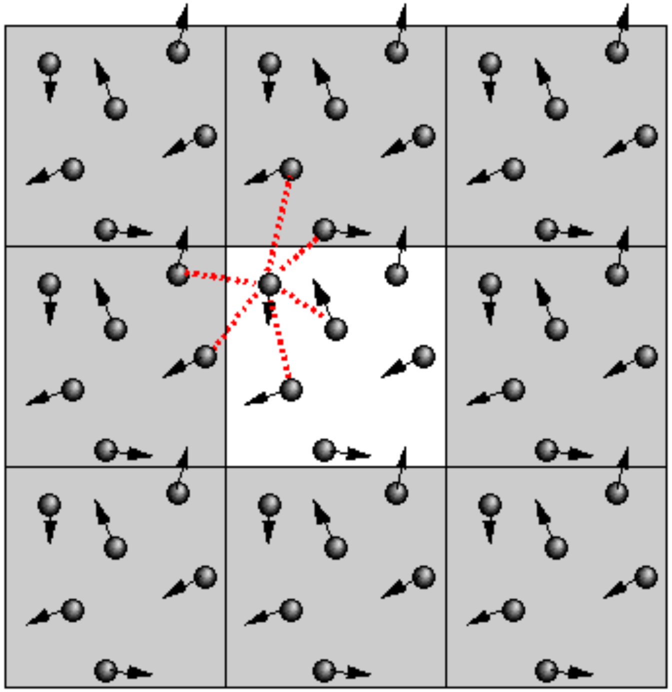
終端抵抗：ターミネーター
と呼ぶことが多い







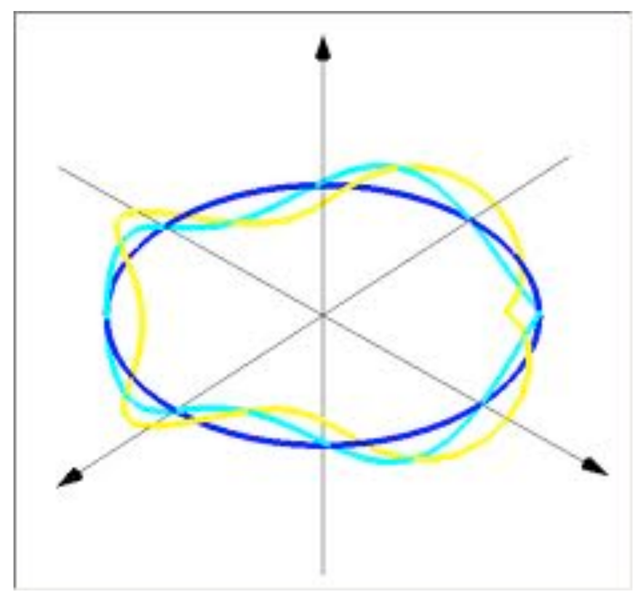
周期境界条件　境界を「てっとりばやく」消してしまう方法



むかい合う境界を「はり合わせて」しまって、境界はないけれども閉じた空間を作る。

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x) = E_n\psi_n(x)$$

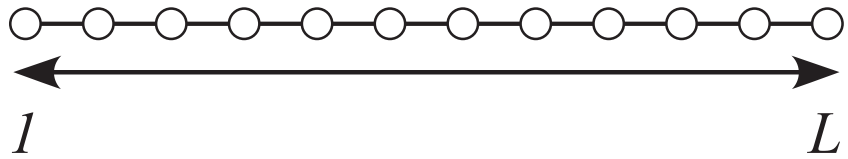
1 次元井戸型ポテンシャルの問題では、x=0 と x=L を同一視して、空間を「輪」にしてしまう。→→



... でも、ちょっと、わざとらしくない？

空間を曲げないと輪にならないよ？？

ようやく、本日の問題設定：



有限な 1 次元の格子上でフェルミ粒子が飛び移る系

$$\mathcal{H}^{(N)} = -t \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

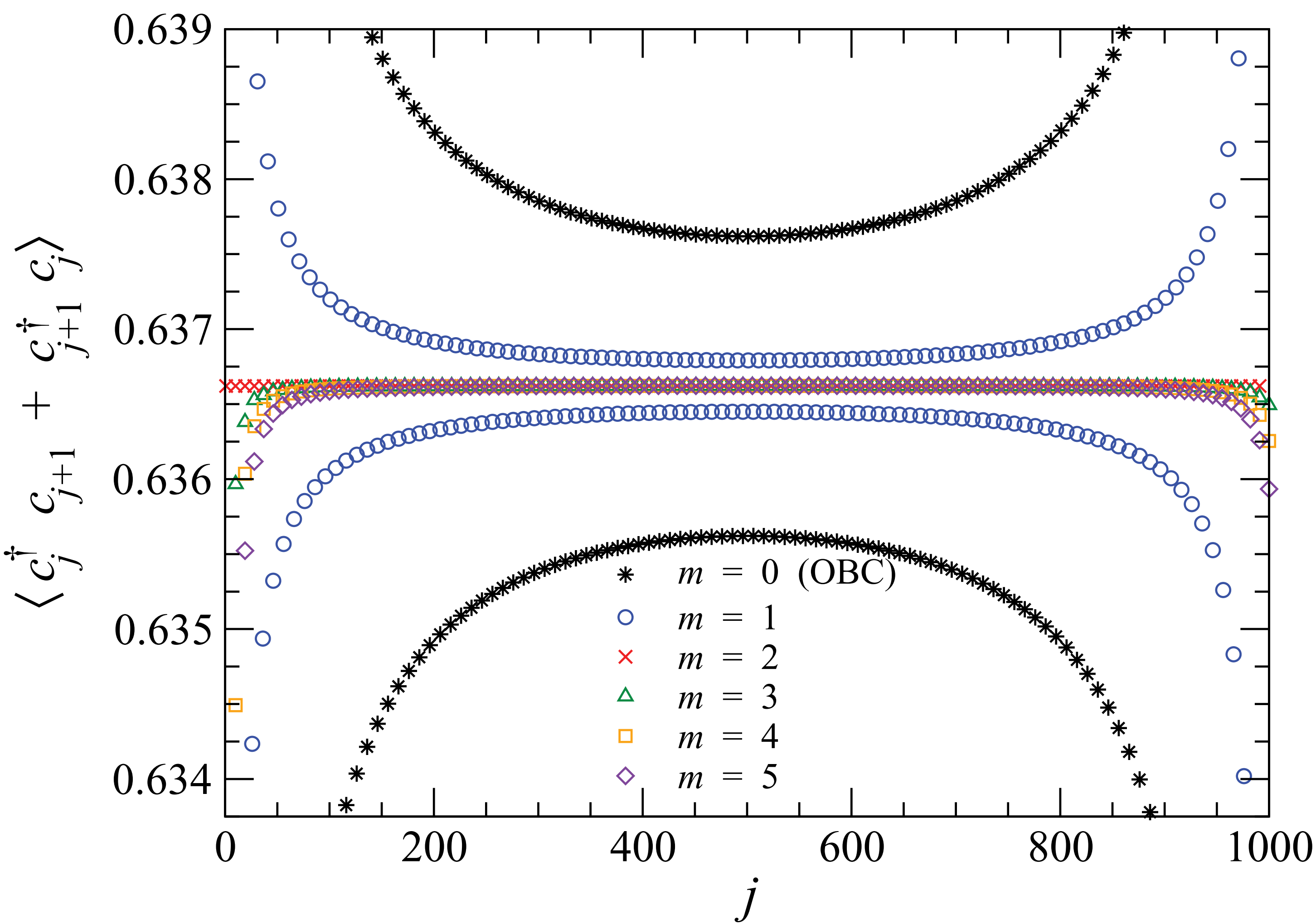
空間が離散的、つまり「並んだ格子点」で表現されている点を除いて、井戸型ポテンシャルと全く同じ問題。この格子上に置かれた粒子は、確率振幅 -t で左右の格子に飛び移り、境界である j = 1 や j = N から外へ、つまり j = 0 や j = N+1 へと出て行くことはない。

粒子を格子点の数の半分まで入れて、物理量を観察してみよう。（フェルミ粒子だから、波動関数はスレーター行列式で与えられる。）

$$\mathcal{H}_{\mathrm{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N} \right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

格子点の半分の数の粒子を放り込んで、再近接相関関数を計算してみると？

m = 2 で、境界効果が「ほとんど」消失してしまった。



失敗談

$$\mathcal{H}_{\rm sine}^{(N)} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N}\right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Andrej Gendiar 氏が m = 2 の場合について集めた計算データを、メールで受け取る際の「ドサクサ」で、私は m = 1 のデータだと思い込んで論文を書いてしまった。↓そして「運悪く」そのまま掲載されてしまった .....

Progress of Theoretical Physics, Vol. 122, No. 4, October 2009

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR,[1,2] Roman KRCMAR[1] and Tomotoshi NISHINO[2,3]

科学者は「正直者」でなければならない。（但し論文を書く時「だけ」）「すんません、m=2 の間違いでした」という報告を書いて、その雑誌に掲載してもらった。

Errata

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR, Roman KRCMAR and Tomotoshi NISHINO

Prog. Theor. Phys. **122** (2009), 953.

(Received December 10, 2009; Revised December 23, 2009)

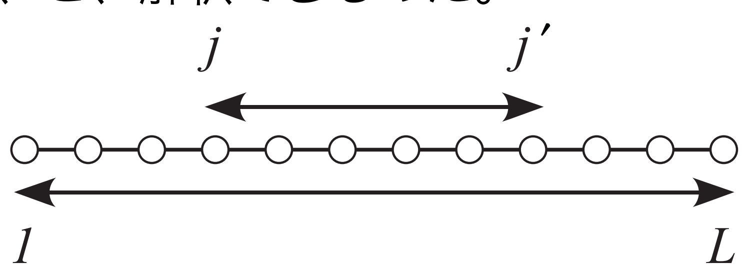
どうして境界効果が消失したの？

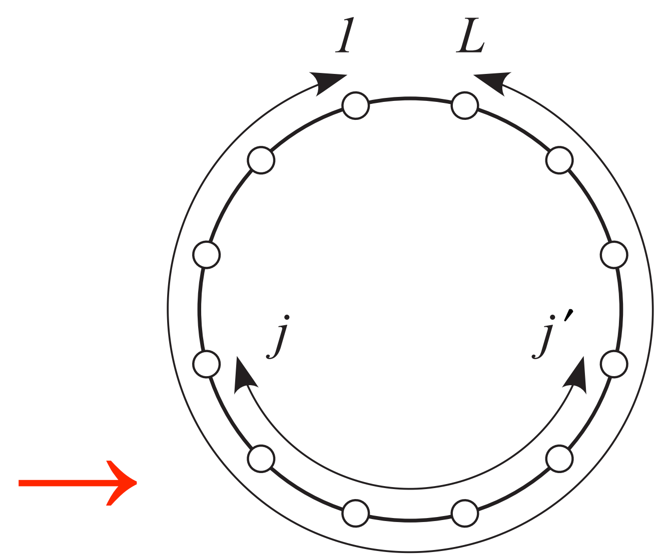
$$\mathcal{H}_{\text{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \left[\sin\left(\frac{j\pi}{N} \right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

基底波動関数を（数値的に）調べてみると、m = 2 の場合の波動関数は「hopping amplitude が一様な系で、周期境界条件を課した場合」の基底波動関数と全く同じであることがわかった。

ここが量子力学の不思議！　←広義には「波動物理学」の不思議

境界へ向けて、飛び移りの大きさを連続的に変化させて行くだけで、系の右端と左端が effective に接続されてしまった、と、解釈できるのだ。





数値計算から数理物理学へ

$$\hat{H}_{\mathrm{S}} = -t \sum_{\ell=1}^{N-1} \sin^2 \frac{\ell\pi}{N} \left(\hat{c}_\ell^\dagger \hat{c}_{\ell+1} + \hat{c}_{\ell+1}^\dagger \hat{c}_\ell \right)$$

... 私は足し算や引き算は苦手だから、ここから
先は計算が得意な方にバトンタッチする ....

Exact ground state of the sine-square deformed XY spin chain

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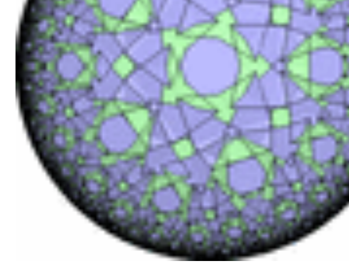
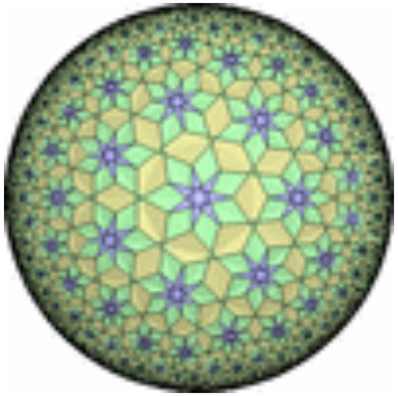
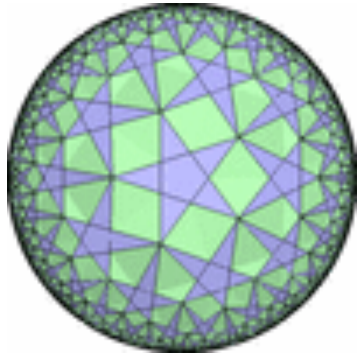
Abstract. We study the sine-square deformed quantum XY chain with open boundary conditions, in which the interaction strength at the position x in the chain of length L is proportional to the function $f_x = \sin^2[\frac{\pi}{L}(x - \frac{1}{2})]$. The model can be mapped onto a free spinless fermion model with site-dependent hopping amplitudes and on-site potentials via the Jordan-Wigner transformation. Although the single-particle eigenstates of this system cannot be obtained in closed form, it is shown that the many-body ground state is identical to that of the uniform XY chain with periodic boundary conditions. This proves a conjecture of Hikihara and Nishino [Hikihara T and Nishino T 2011 *Phys. Rev. B* **83** 060414(R)] based on numerical evidence.

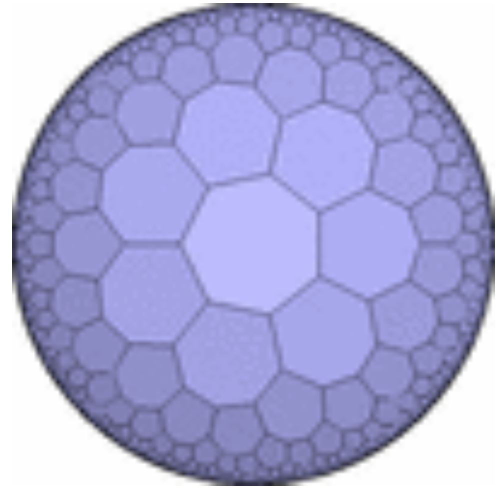
証明されちゃった!！　但し、背景の「代数」は未解決なまま ...

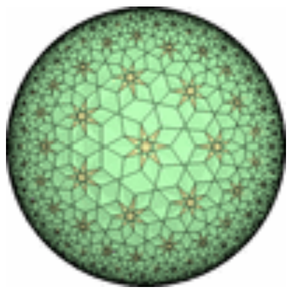
柳の下にはドジョウが百匹：あらゆる等質空間へ!!!

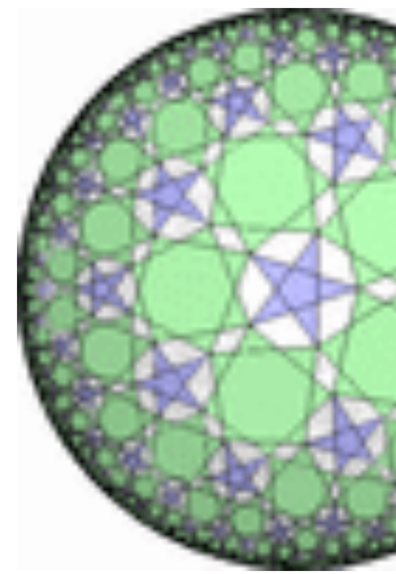


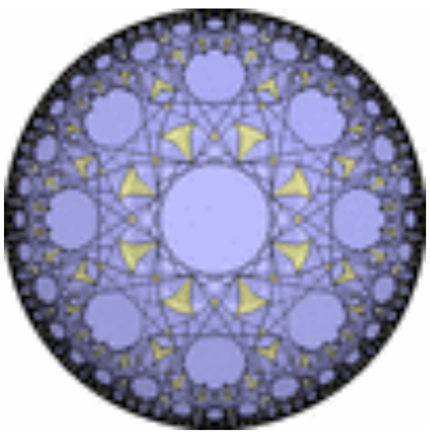


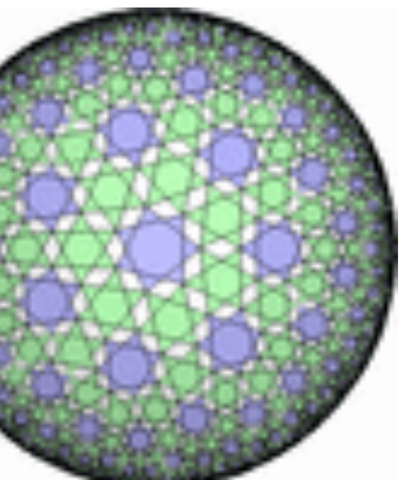










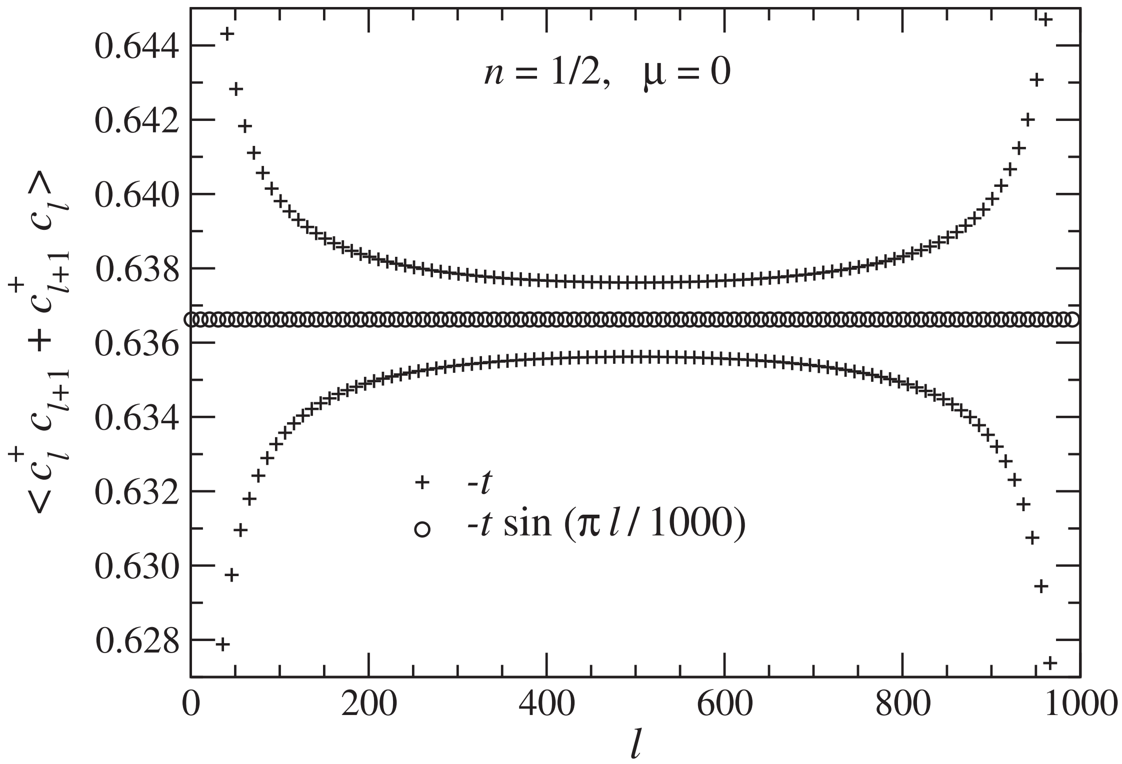


まあ、今日はこの辺で
失礼します。

A Generalization: Spherical Deformation

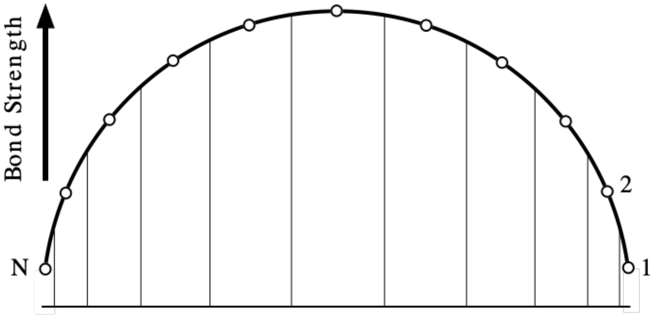
N-site tight binding Hamiltonian

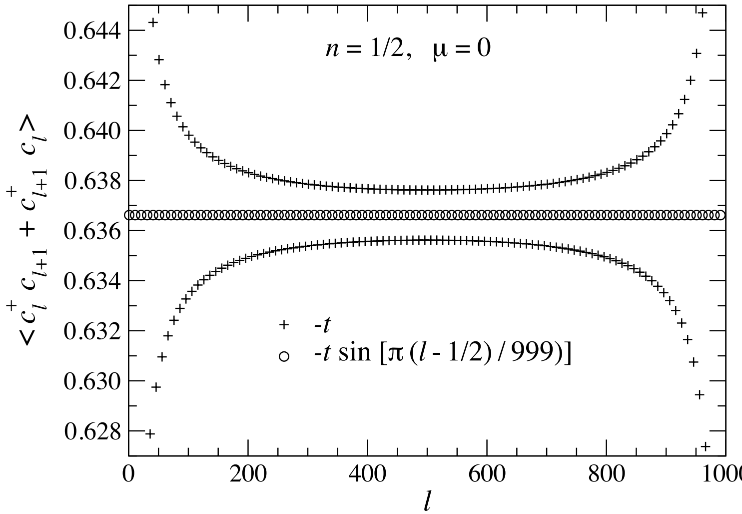
$$\hat{H}_{\mathrm{S}} = -t \sum_{\ell=1}^{N-1} \sin^2 \frac{\ell\pi}{N} \left(\hat{c}_\ell^\dagger \hat{c}_{\ell+1} + \hat{c}_{\ell+1}^\dagger \hat{c}_\ell \right)$$

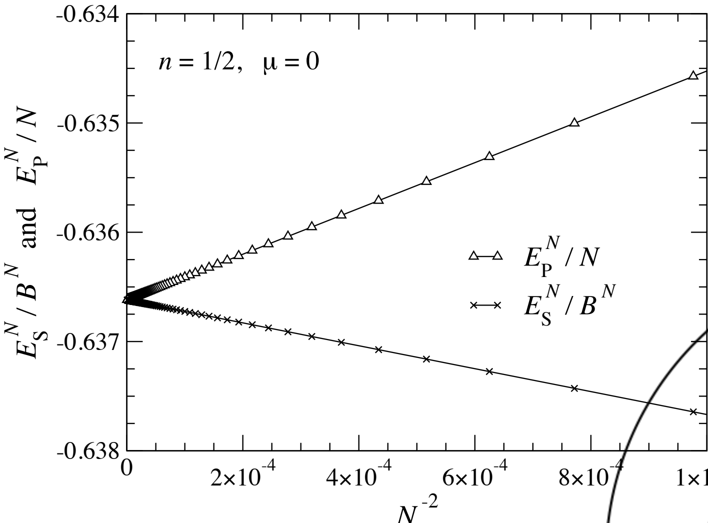


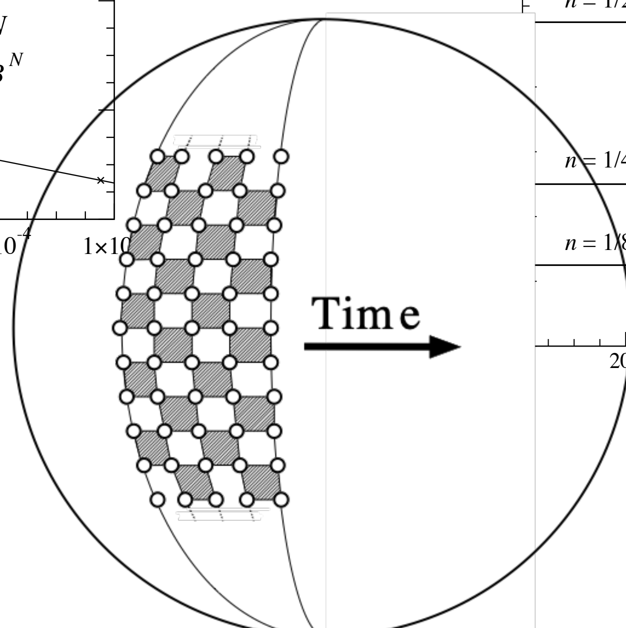
Boundary effect on the bond energy disappears completely!

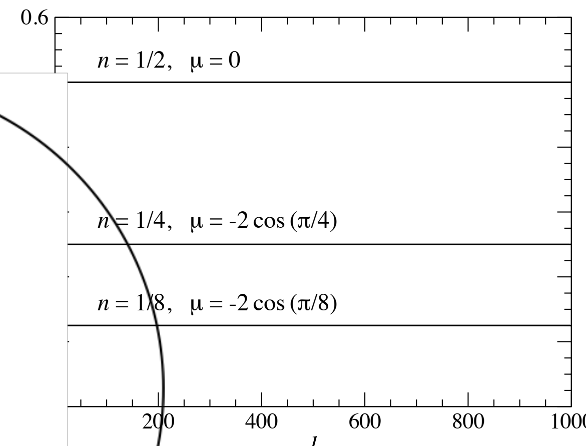
A system under Open Boundary Condition gives data as efficient as those under Periodic Boundary Condition, under the spherical deformation.


Bond Strength
N
1
2


n = 1/2, μ = 0
<c_l^+ c_{l+1} + c_{l+1}^+ c_l>
+ -t
○ -t sin [π (l - 1/2) / 999)]
0.644
0.642
0.640
0.638
0.636
0.634
0.632
0.630
0.628
0
200
400
600
800
1000
l


n = 1/2, μ = 0
E_S^N / B^N and E_P^N / N
E_P^N / N
E_S^N / B^N
-0.634
-0.635
-0.636
-0.637
-0.638
0
2×10^-4
4×10^-4
6×10^-4
8×10^-4
N^-2


Time


0.6
n = 1/2, μ = 0
n = 1/4, μ = -2 cos (π/4)
n = 1/8, μ = -2 cos (π/8)
200
400
600
800
1000
l