- Charge/Spin density wave, commensurate or incommensurate

- ex. Axial Next Nearest Neighbor Ising (ANNNI) model

\[ \mathcal{H} = -J_1 \sum_{i,j} \sigma_{i,j} (\sigma_{i+1,j} + \sigma_{i,j+1}) - J_2 \sum_{i,j} \sigma_{i,j} \sigma_{i+2,j} \]
Energy Scale Deformation

* There is a modulated Hamiltonian whose ground state is uniform.

- empty state of any Fermionic system (too trivial!)
- (modulated/inhomogeneous) AKLT Hamiltonian
  
  since $H = \text{sum of projectors}$, and pre factor can be arbitral

- Slow energy scale modulation would not affect a gapped ground state
  
  if the modulation is slow enough (or gap is wide enough)

- Exponential Deformation (Wilson, ..., Okunishi)

  wilson lattice \textbf{arXiv:1001.2594}

  $$\mathcal{H}_\lambda = \sum_{n=1}^{N-1} e^{\lambda n} (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1})$$

  general framework \textbf{arXiv:cond-mat/0702581}

  $$H_N(\Lambda) = \sum_{n=1}^{N-1} \Lambda^{N-n-1} h_{n,n+1},$$
a classical counterpart: Hyperbolic Lattice

Ising model on Hyperbolic Lattice

- there is ferro-para phase transition
- always off critical
- row-to-row transfer matrix can be defined
- is it possible to find out the corresponding quantum Hamiltonian? (I have no answer)

probably, in anisotropic limit (how to define this limit?), one reaches the hyperbolic deformation. \[H^{\cosh}(\lambda) = \frac{1}{2}[H^{\exp}(\lambda) + H^{\exp}(-\lambda)]\]

\[= \sum_{j=-N}^{N} \cosh j\lambda \ h_{j,j+1}.\]

ground-state is uniform, except for the edge state, as it was observed in the case of exp. deformation.
a path to “spherical” deformation

* Corner Hamiltonian ~ Entanglement Hamiltonian

- Okunishi proposed a quantum counterpart of CTMRG

\[
K_N = \sum_{n=1}^{N-1} n h_{n,n+1}, \quad \text{cond-mat/0507195}
\]

- Hyperbolic “deformation” can be considered

\[
H^{\sinh}(\lambda) = \sum_{j=-N}^{N} \sinh j \lambda \ h_{j,j+1}, \quad \text{arXiv:0808.3858}
\]

* History in physics suggests the generalization to trigonometric deformations

\[
H_{\text{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell) \ h_{\ell,\ell+1} \quad \text{arXiv:0810.0622}
\]

... well, the prototype was “cosine deformation”, and not squared. How can one use the deformation? (I don’t know.)
最近接格子点間の「相関関数」を求めてみる。N=1000 サイドの系での計算結果は「境界効果」であるフリーデル振動が、内部まで浸透していることがわかる。(金属表面で電子密度が振動するのも同じようなもの)
Smooth Boundary Conditions

斉飛び振幅 -t を、系の両端で小さくすれば、上手く「ターミネート」できるのではないか？

PHYSICAL REVIEW LETTERS

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Smooth Boundary Conditions for Quantum Lattice Systems

M. Vekić and S. R. White

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(Received 1 September 1993)

We introduce a new type of boundary conditions, smooth boundary conditions, for numerical studies of quantum lattice systems. In a number of circumstances, these boundary conditions have substantially smaller finite-size effects than periodic or open boundary conditions. They can be applied to nearly any short-ranged Hamiltonian system in any dimensionality and within almost any type of numerical approach.

PACS numbers: 02.70.-c, 05.30.Fk, 75.10.Jm
White の成果

飛び移り振幅 -t の、系の両端でのスムージング関数

化学ポテンシャル変化に対する粒子密度の変化
レンズのコーティングもまた同じ

← 望遠鏡の善し悪しは、対物レンズのコーティングを見ると、おおよそ推測できることが多い。
（粗悪品は値段の割に口径が大！）
a path to “spherical” deformation

* Corner Hamiltonian ~ Entanglement Hamiltonian

- Okunishi proposed a quantum counterpart of CTMRG

\[ K_N = \sum_{n=1}^{N-1} n h_{n,n+1}, \quad \text{cond-mat/0507195} \]

- Hyperbolic “deformation” can be considered

\[ H^{\sinh}(\lambda) = \sum_{j=-N}^{N} \sinh j\lambda \ h_{j,j+1}, \quad \text{arXiv:0808.3858} \]

* History in physics suggests the generalization to trigonometric deformations

\[ H_{\text{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell) \ h_{\ell,\ell+1} \quad \text{arXiv:0810.0622} \]

... well, the prototype was “cosine deformation”, and not squared. How can one use the deformation? (I don’t know.)
Spherical Deformation for One-dimensional Quantum Systems

Andrey Gendiar, Roman Krcmar, Tomotoshi Nishino

(Submitted on 3 Oct 2008 (v1), last revised 27 Dec 2010 (this version, v6))

\[ H_S^N = -t \sum_{\ell=-N/2}^{N/2-2} \cos \left( \frac{\ell + 1}{N-1} \pi \right) \left( c_{\ell+1} c_{\ell+1} + c_{\ell+1} c_{\ell} \right) \]

... finally we reach sin^2 form, ... almost ACCIDENTALLY


Errata

Spherical Deformation for one-dimensional Quantum Systems

Andrey GENDIAR, Roman KRCMAR, and Tomotoshi NISHINO


In the article we have published, we studied the finite-size correction to the energy per site \( E^N / N \) for the spherically deformed free fermion lattice, whose Hamiltonian is given by

\[
\hat{H}_S^{(n)} = \sum_{\ell=1}^{N-1} \left[ \sin \frac{\ell \pi}{N} \right]^n \left( -t \hat{c}_\ell \hat{c}_{\ell+1} - t \hat{c}_{\ell+1} \hat{c}_\ell - \mu \frac{\hat{c}_\ell \hat{c}_\ell + \hat{c}_{\ell+1} \hat{c}_{\ell+1}}{2} \right)
\] (1)
What happened?

- I visited Aachen, to discuss with Andrej Gendiär in 2008.

  … we considered a way of reducing the boundary effect in 1D chain.

The following picture came up, though I do not understand what it is even now. (open problem)
a sphere has no border

let us focus on the width of each piece of paper.

\[
\mathcal{H}_{\text{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \sin \left( \frac{j\pi}{N} \right)^m \left( c_j^{\dagger} c_{j+1} + c_{j+1}^{\dagger} c_j \right)
\]

Major contribution came from Andrej Gendiar
What happened?
- I visited Aachen, to discuss with Andrej Gendiär in 2008.

... we considered a way of reducing the boundary effect in 1D chain.

The following picture came up, though I do not understand what it is even now. (open problem)

any way, we checked the “cosine deformation” on the free fermion lattice, and confirmed that it reduces the boundary effect.

\[ H_{\text{Sph.}} = \sum_{\ell=-N/2}^{N/2-1} \cos(a\ell) h_{\ell,\ell+1} \]

We report the result as [v1] of arXiv:0810.0622

Referee pointed that the boundary effect is reduced, but still there is.

- Andrej proposed to consider \( \cos^n \) also, since the function falls to 0 MORE SMOOTHLY than \( \cos^1 \).

- I denied Andrej’s proposal, since \( \cos^n \) contradict the above SPHERE.
What happened?
- I visited Aachen, to discuss with Andrej Gendiar in 2008.
  ... we considered a way of reducing the boundary effect in 1D chain.

The following picture came up, though I do not understand what it is even now. (open problem)

![Diagram](image)

**Figure 2.** Expectation value $\langle c^\dagger_\ell c_{\ell+1} + c^\dagger_{\ell+1} c_\ell \rangle$ of the spherically deformed lattice Fermion model when $N = 400$. For comparison, we also plot the same expectation value for the undeformed case.

3. Spherical deformation
Consider an $N$-site open boundary system described by the Hamiltonian

$$H_N^{SS} = -\frac{tN}{2} - 2\sum_{\ell=-N/2}^{N/2-1} \cos \left( \frac{\ell+1}{N-1} \pi \right) (c^\dagger_\ell c_{\ell+1} + c^\dagger_{\ell+1} c_\ell)$$

(18)

If compared with the undeformed Hamiltonian $H_N^{SO}$ in Eq. (7), the strength of the hopping term is scaled by the factor $A_\ell = \cos \left[\frac{(\ell+1)\pi}{N-1}\right]$, which decreases towards the system boundary. Figure 1 schematically shows such a scale. For a reason which we discuss in the next section, we call the deformation from $H_N^{SO}$ to $H_N^{SS}$ as the spherical deformation. So far we have not obtained an analytic form of the one-particle wave function $\psi_N^{SS}; m$ and the corresponding eigenvalue $\varepsilon_N^{SS}; m$. Thus, we calculate them numerically by diagonalizing $H_N^{SS}$ for the case where there is a particle in the system, and obtain the ground state energy and expectation value $\langle c^\dagger_\ell c_{\ell+1} + c^\dagger_{\ell+1} c_\ell \rangle$ at half filling.
What happened?  - Andrej was right, and there is one another side story.


In the article we have published, we studied the finite-size correction to the energy per site \( E^N/N \) for the spherically deformed free fermion lattice, whose Hamiltonian is given by

\[
\hat{H}_S^{(n)} = \sum_{\ell=1}^{N-1} \left[ \sin \frac{\ell \pi}{N} \right]^n \left( -t \hat{c}^\dagger_\ell \hat{c}_{\ell+1} - t \hat{c}^\dagger_{\ell+1} \hat{c}_\ell - \mu \frac{\hat{c}^\dagger_\ell \hat{c}_\ell + \hat{c}^\dagger_{\ell+1} \hat{c}_{\ell+1}}{2} \right) \tag{1}
\]

for the case \( n = 1 \). While we proceeded to a further study on the spherical deformation, we noticed the data shown in Figs. 2-7 were incorrect, and these figures corresponded to the Hamiltonian for the case \( n = 2 \). This error happened due to a very primitive confusion in the file name of computational source codes, and we misused the data with \( n = 2 \), instead of \( n = 1 \). We show appropriate data for the typical case \( \mu = 0 \), which corresponds to the half filling.

Fig. 1. Bond correlations at half filling calculated for \( \hat{H}_S^{(n)} \) with \( n = 0, 1, \) and 2.

Fig. 2. Finite-size corrections to the energy.
Home Works (Conjectures)

Extension to higher dimensional system

- It is always possible to consider Hyperbolic lattice or deformation.
- Slowest modulation on N-dimensional sphere would be an extension of SSD.

Trotter decomposition

- What is the right Trotter decomposition between curved surface with constant curvature and corresponding quantum (lattice) system.

Fuzzy space

- How does non commutable space can be deformed in the manner of SSD?
What is the most appropriate form for the high field magnet?

Liu et al. [arXiv:1907.03539]

Spherical coil? Hyperbolic helical coil?
Do find something rectangular/cylindrical to fill this space.

try to find on SNS.
Do find something rectangular/cylindrical

You are looking at rectangular screen.

u phone, also.
以下、付録
境界条件 (Boundary Condition) というもの

同じ水面でも、その性質は容れ物によって大きく変化する。

注) 文字が現れるのは一瞬だけ→
境界の効果は「反射」として現れる

→ 反射を減じて「無限を演出」したくなることもある

例えば水道では（通称）蛇口という境界があって、急に閉じると強い圧力波を発生させてしまう。

そこで、こんなものが裏で使われている、ことがある。
電気回路（や音響回路など）のインピーダンス整合も
境界からの信号反射を減じるための工夫である。

昔なつかしい SCSI のターミネーター

終端抵抗：ターミネーターと呼ぶことが多い

密閉型デバイス

T-Port タップ

PowerTap タップ

DevicePort タップ（8 ポート）

DeviceBox タップ（4 ポート）

太線ケーブル

密閉型デバイス

密閉型デバイス

細線ケーブル

T-Port タップ

開放型デバイス

開放型デバイス
周期境界条件 境界を「てっとりばやく」消してしまう方法

むかい合う境界を「はり合わせて」しまって、境界はないとすればでも閉じた空間を作る。

\[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) = E_n \psi_n(x)\]

1次元井戸型ポテンシャルの問題では、x=0 と x=L を同一視して、空間を「輪」にしてしまう。

... でも、ちょっと、わざとらしくない？ 空間を曲げないと輪にならないよ？？
有限な1次元の格子上でフェルミ粒子が飛び移る系

$$H^{(N)} = -t \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

空間が離散的、つまり「並んだ格子点」で表現されている点を除いて、井戸型ポテンシャルと全く同じ問題。この格子上に置かれた粒子は、確率振幅$t$で左右の格子に飛び移り、境界である$j=1$や$j=N$から外へ、つまり$j=0$や$j=N+1$へと出て行くことはない。

粒子を格子点の数の半分まで入れて、物理量を観察してみよう。（フェルミ粒子だから、波動関数はスレーター行列式で与えられる。）
where the leading correction is of the order of the definition of the spectrum of which is the sum of the deformation factors over the entire system with OBCs, and PBCs, one finds the difference between Eqs. (9), (10). We refer to the uniformity (the translation invariance) of the bond correlation function for both cases at half filling, where the correction is present when the particle number might be related to the suppression of finite-size effects in one-dimensional systems.

\[ \mathcal{H}_{\text{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \left[ \sin \left( \frac{j \pi}{N} \right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \]

格子点の半分の数の粒子を放り込んで、再近接相関関数を計算してみると？

\[ m = 2 \text{ で、境界効果が「ほとんど」消失してしまった。} \]

![Graph showing the asymptotic behavior of energy per bond for different cases.](image)
Andrej Gendiar氏が$m=2$の場合について集めた計算データを、メールで受け取る際の「ドサクサ」で、私は$m=1$のデータだと思い込んで論文を書いてしまった。↓そして「運悪く」そのまま掲載されてしまった.....

Progress of Theoretical Physics, Vol. 122, No. 4, October 2009

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR, Roman KRCMAR and Tomotoshi NISHINO

科学者は「正直者」でなければならない。（但し論文を書く時「だけ」）「すみません、m=2の間違いでした」という報告を書いて、その雑誌に掲載してもらった。

Errata

Spherical Deformation for One-Dimensional Quantum Systems

Andrej GENDIAR, Roman KRCMAR and Tomotoshi NISHINO


(Received December 10, 2009; Revised December 23, 2009)
どうして境界効果が消失したの？

$$
\mathcal{H}_{\text{sine}}^{(N)} = -t \sum_{j=1}^{N-1} \left[ \sin \left( \frac{j \pi}{N} \right) \right]^m (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)
$$

基底波動関数を（数値的に）調べてみると、$$m = 2$$ の場合の波動関数は「hopping amplitude が一様な系で、周期境界条件を課した場合」の基底波動関数と全く同じであることがわかった。

ここが量子力学の不思議！ ←広義には「波動物理学」の不思議

境界へ向けて、飛び移りの大きさを連続的に変化させて行くだけで、系の右端と左端が effective に接続されてしまった、と、解釈できるのだ。
Exact ground state of the sine-square deformed XY spin chain

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Abstract. We study the sine-square deformed quantum XY chain with open boundary conditions, in which the interaction strength at the position \(x\) in the chain of length \(L\) is proportional to the function \(f_x = \sin^2\left[\frac{\pi}{L}(x - \frac{1}{2})\right]\). The model can be mapped onto a free spinless fermion model with site-dependent hopping amplitudes and on-site potentials via the Jordan-Wigner transformation. Although the single-particle eigenstates of this system cannot be obtained in closed form, it is shown that the many-body ground state is identical to that of the uniform XY chain with periodic boundary conditions. This proves a conjecture of Hikihara and Nishino [Hikihara T and Nishino T 2011 Phys. Rev. B 83 060414(R)] based on numerical evidence.
柳の下にはドジョウが百匹：あらゆる等質空間へ!!!

まあ、今日はこの辺で失礼します。
A Generalization: Spherical Deformation

N-site tight binding Hamiltonian

\[ \hat{H}_S = -t \sum_{\ell=1}^{N-1} \sin \frac{2\ell\pi}{N} \left( \hat{c}_\ell \hat{c}_{\ell+1} + \hat{c}_\ell^\dagger \hat{c}_{\ell+1}^\dagger \right) \]

A system under Open Boundary Condition gives data as efficient as those under Periodic Boundary Condition, under the spherical deformation.