Perspectives from Sine-square deformation on conformal field theories

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arXiv:1504.00138
1602.01190
1712.09823
1904.12414

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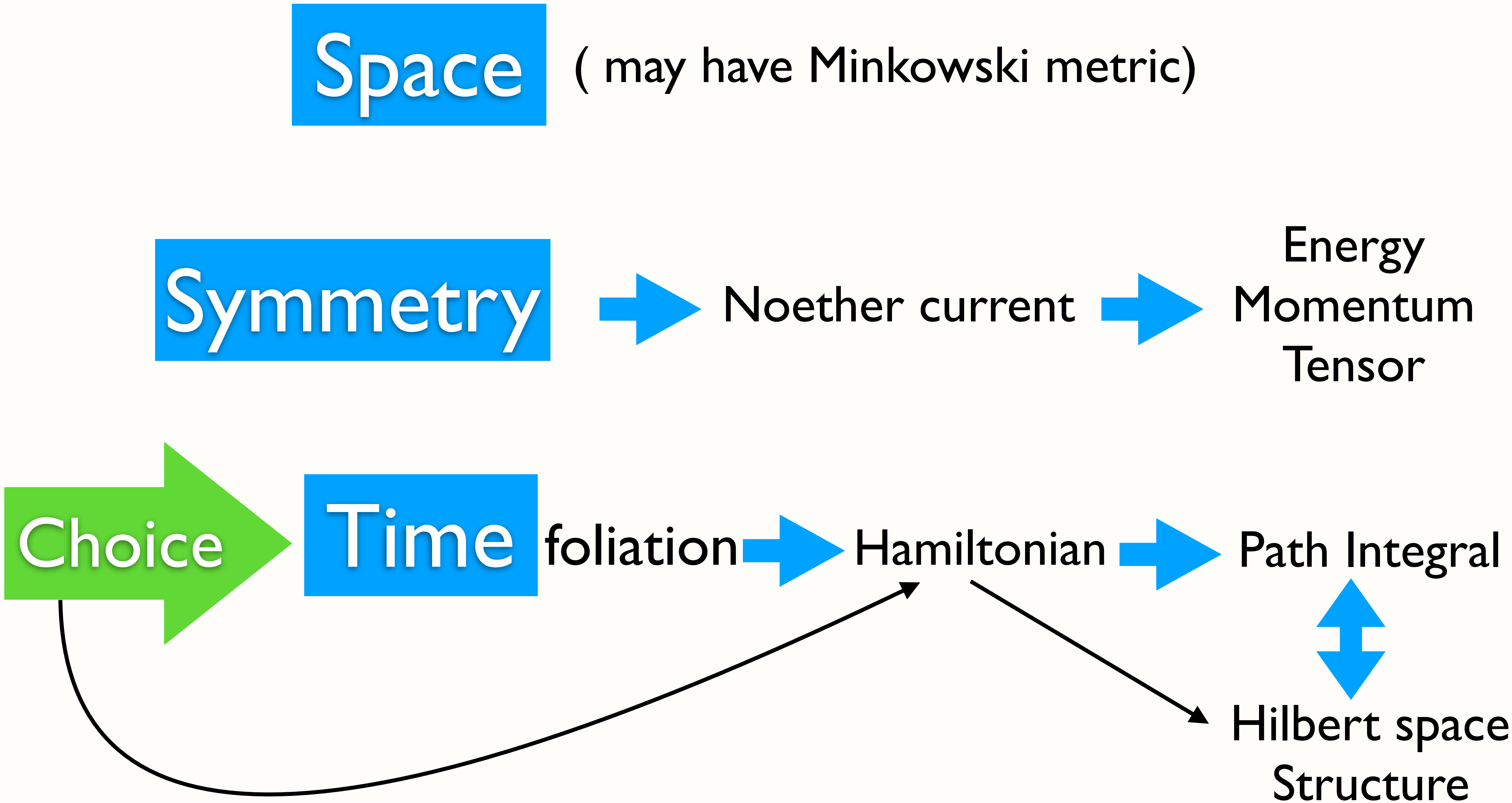

Space
(may have Minkowski metric)
Symmetry
Noether current
Energy Momentum Tensor
Choice
Time
foliation
Hamiltonian
Path Integral
Hilbert space Structure

Principle for

Choice → Time foliation

↓ respect

Symmetry → Conservation of "Energy" guaranteed

↓

Unitarity ↔ Hilbert space Structure

What

Symmetry

Conformal symmetry

$$\cup$$

$$SL(2,\mathbb{R})\left(\times SL(2,\mathbb{R})\right)$$

Choice

Classified by

Casimir invariant

What

Symmetry Conformal symmetry

Space accomodates $SL(2,\mathbb{R})\left(\times SL(2,\mathbb{R})\right)$

$\mathbb{S}^2 \quad \mathbb{H}^2$

$\mathbb{AS}_3$ → Ryu-san's talk

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$

2) $\mathcal{H} = L_0 - \dfrac{L_1 + L_{-1}}{2} + \bar{L}_0 - \dfrac{\bar{L}_1 + \bar{L}_{-1}}{2}$

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

Casimir

$$c^{(2)} \equiv \left(x^{(0)}\right)^2 - 4x^{(1)}x^{(-1)} \Longleftrightarrow x^{(0)}L_0 + x^{(1)}L_1 + x^{(-1)}L_{-1}$$

$$\Big\| \qquad\qquad \Big\downarrow \text{adjoint action}$$

$$\left(x'^{(0)}\right)^2 - 4x'^{(1)}x'^{(-1)} \qquad x'^{(0)}L_0 + x'^{(1)}L_1 + x'^{(-1)}L_{-1}$$

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$

$c^{(2)} > 0$

2) $\mathcal{H} = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2}$

$c^{(2)} = 0$

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$c^{(2)} < 0$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

Three choices

1) $\mathcal{H} = L_0 + \bar{L}_0$ radial quantization

2) $$\mathcal{H} = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2}$$

dipolar quantization Continuous Virasoro algebra

Sine-square Deformation

3) $\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$

$\mathcal{H}$: Hamiltonian L_n :Virasoro generator

$$\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$$

$$H[f] = \int dx\, f(x)\, \mathcal{H}(x).$$

Envelope function

TABLE I. Summary of conformal maps and deformed evolution operators discussed in the main te

	Conformal map	"Time"	"Space"	Envelope function
Angular quantization	$w = \ln z$	v	u	$f(x) = x$
Radial quantization	$w = \ln z$	u	v	$f(s) = \frac{1}{L}$
Entanglement Hamiltonian	$w = \ln \frac{(z+R)}{(z-R)}$	v	u	$f(x) = \frac{(x-R)(x+R)}{2R}$
Regularized SSD (rSSD)	$w = \ln \frac{(z+R)}{(z-R)}$	u	v	$f(s) = \cos \frac{2\pi s}{L} + \cosh u_0$
Sine-square deformation (SSD)	$w = \frac{1}{z}$	u	v	$f(s) = \sin^2 \frac{\pi s}{L}$
Square root deformation (SRD)	$z = \sin w$	v	u	$f(x) = \sqrt{x^2 - R^2}$

Entanglement

TT, arXiv: 1904.12414

$$\mathcal{H} = L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}$$

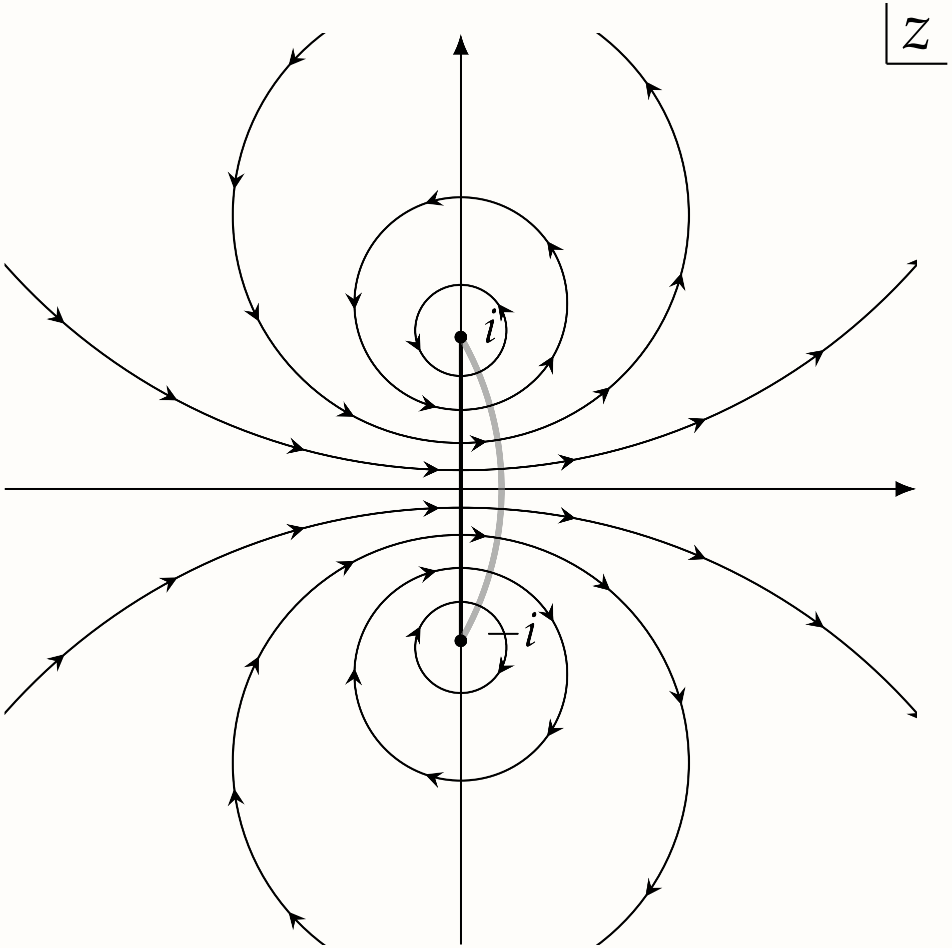
$$g(z) = z^2 + 1$$

$$-\frac{\partial}{\partial t} = -g(z)\frac{\partial}{\partial z} - g(\bar{z})\frac{\partial}{\partial \bar{z}}$$

Entanglement

TT, arXiv: 1904.12414

$$-\frac{\partial}{\partial t} = -z^2\frac{\partial}{\partial z} - \frac{\partial}{\partial z} - \bar{z}^2\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial \bar{z}}$$



Entanglement

TT, arXiv: 1904.12414

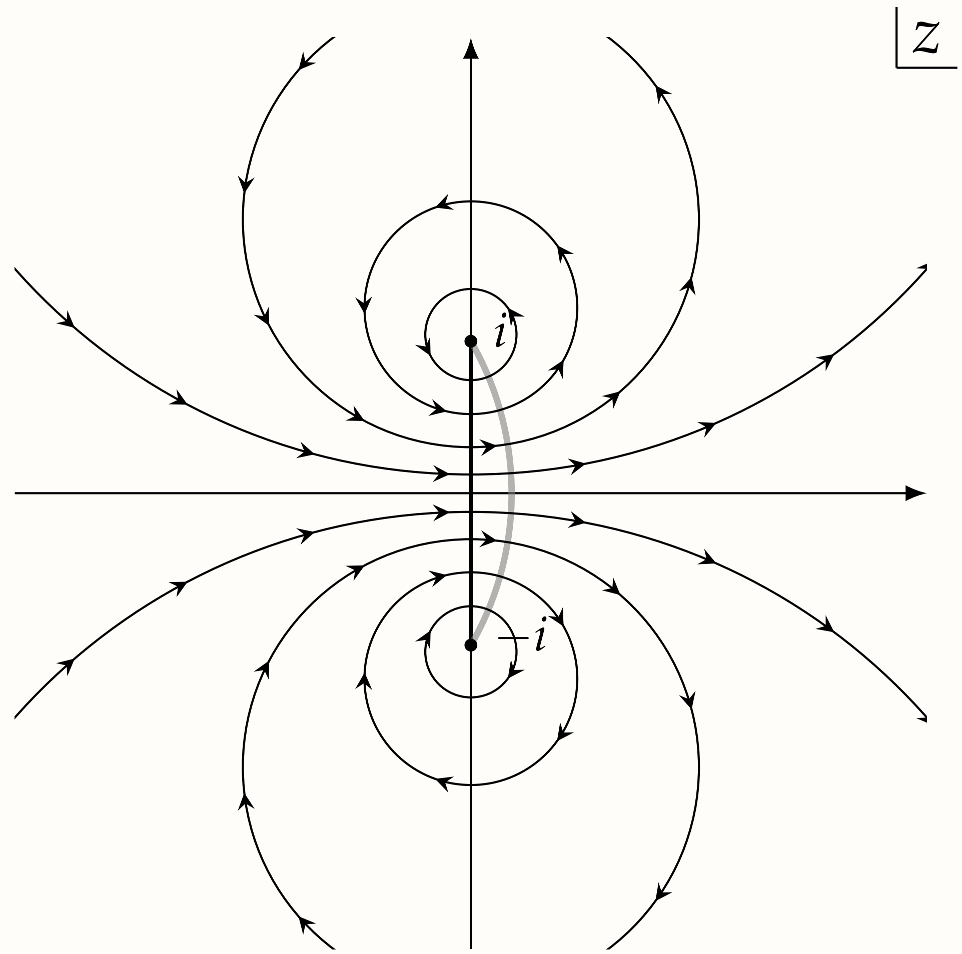
$$\boxed{t + is = \arctan z}$$

$$f_\kappa = \exp(\kappa \int^z \frac{dz}{z^2+1})$$

$$= \left(\frac{i+z}{i-z}\right)^{-\frac{i}{2}\kappa} = e^{\kappa \arctan z}$$

$$\mathfrak{l}_\kappa = -(z^2+1)e^{\kappa \arctan z}\frac{\partial}{\partial z}$$

$$[\mathfrak{l}_\kappa, \mathfrak{l}_{\kappa'}] = (\kappa' - \kappa)g(z)f_\kappa(z)f_{\kappa'}(z)\frac{\partial}{\partial z} = (\kappa - \kappa')\mathfrak{l}_{\kappa+\kappa'}$$



Entanglement

TT, arXiv: 1904.12414

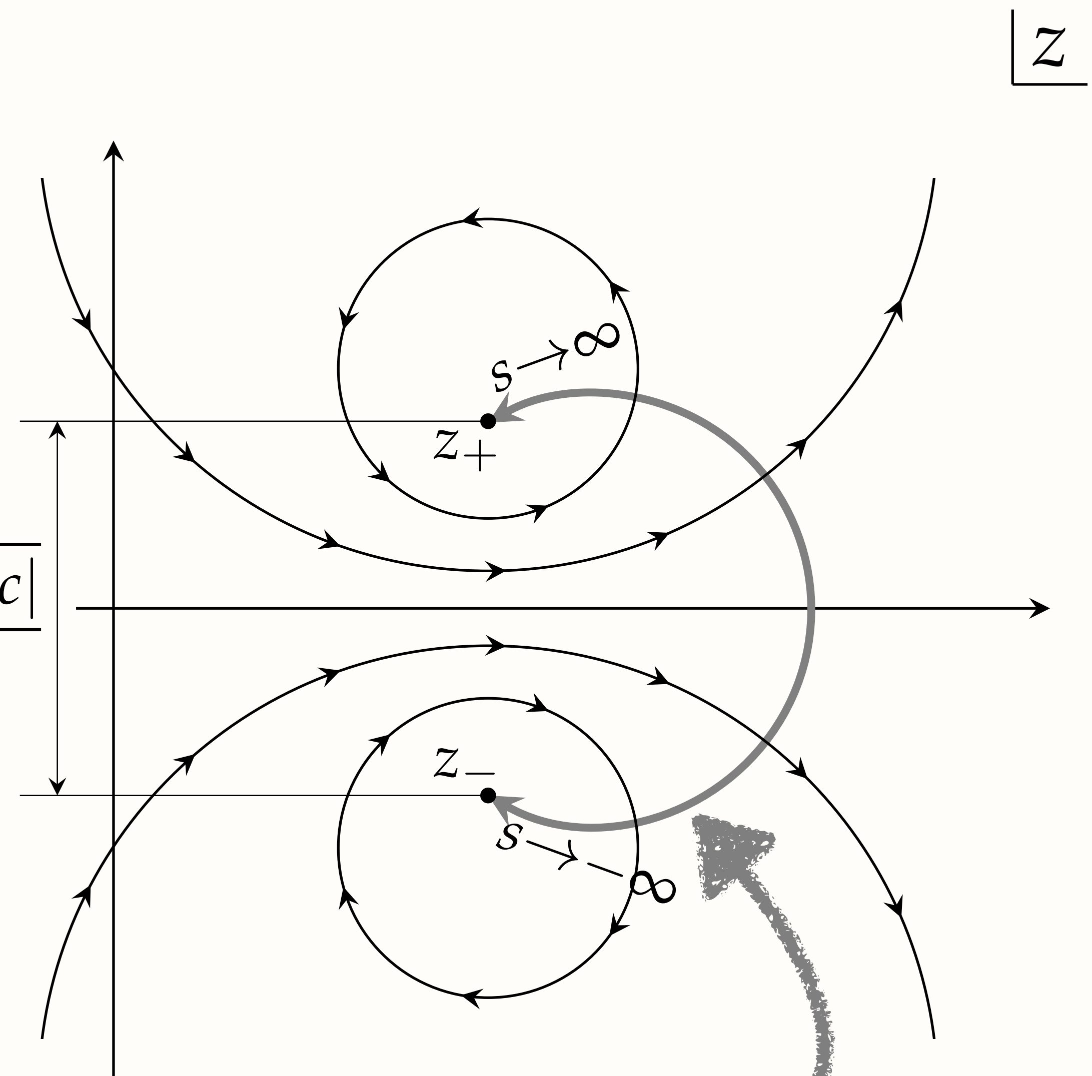
$$g(z) = az^2 + bz + c,$$

$$b^2 - 4ac < 0$$

$$L \equiv \frac{\sqrt{|b^2 - 4ac|}}{a}$$

$$f_\kappa = \exp(\kappa \int^z \frac{dz}{g(z)})$$

$$\mathcal{L}_\kappa \equiv \int_C dz g(z) f_\kappa(z) T(z)$$



Entanglement

TT, arXiv: 1904.12414

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$

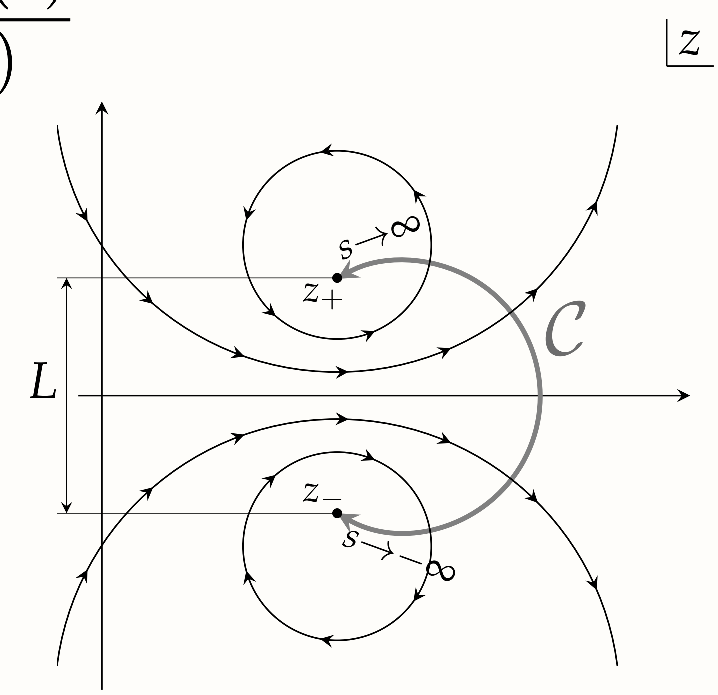
$$\mathrm{CI}[\kappa|\kappa'] \equiv \int_{\mathcal{C}} \frac{dz}{2\pi i}\left\{ g\frac{\partial^3 g}{\partial z^3} + \kappa\left(2\frac{\partial^2 g}{\partial z^2} - \frac{1}{g}\left(\frac{\partial g}{\partial z}\right)^2\right) + \frac{\kappa^3}{g}\right\} f_{\kappa+\kappa'}(z)$$

$$= \left(-(b^2 - 4ac)\kappa + \kappa^3\right)\int_{\mathcal{C}} \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)}$$

Jacobi identity

$$\mathrm{CI}[\kappa|\kappa'] = 0$$

$$\kappa + \kappa' \neq 0$$



Entanglement

TT, arXiv: 1904.12414

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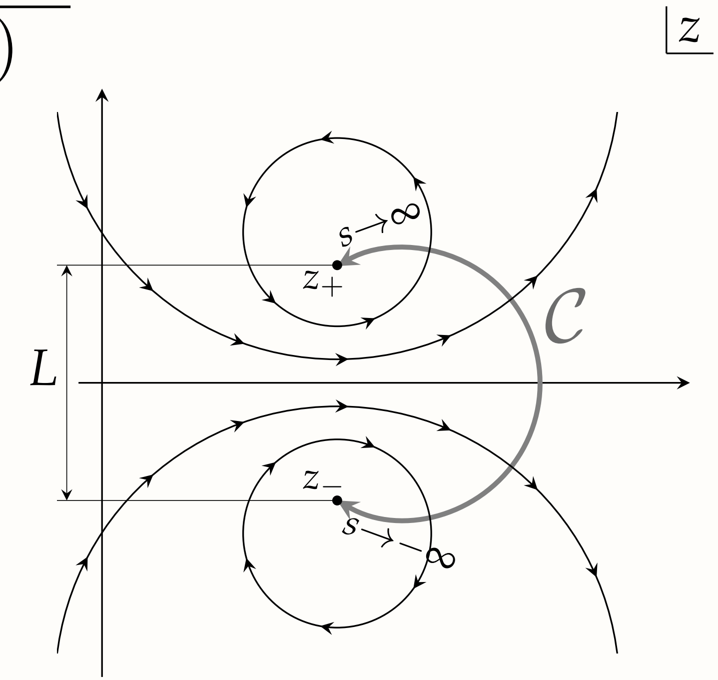
$$= \left(-(b^2-4ac)\kappa + \kappa^3\right)\int_C \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)}$$

$$\int_C \frac{dz}{2\pi i}\frac{f_{\kappa+\kappa'}(z)}{g(z)} = \int_C \frac{df_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}$$

$$= \left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial C}$$



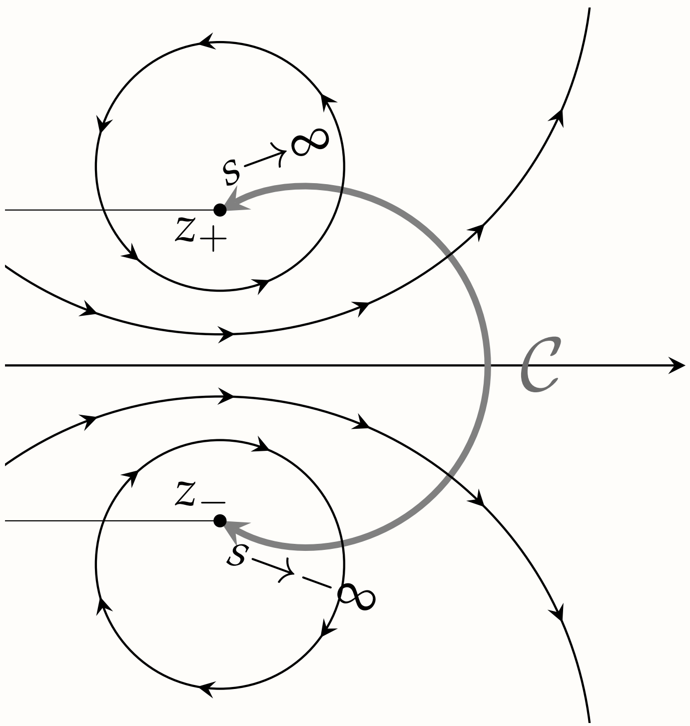
Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\mathrm{CI}[\kappa|\kappa']$$

$\lfloor z$

$$\left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial\mathcal{C}}$$

$$= \frac{1}{2\pi i(\kappa+\kappa')}\left(f_{\kappa+\kappa'}(z_+) - f_{\kappa+\kappa'}(z_-)\right)$$



$$f_\kappa(z) = \exp\left(\kappa\int^z \frac{dz}{a(z-z_+)(z-z_-)}\right) = \exp\left(\frac{\kappa}{a(z_+-z_-)}\ln\left(\frac{z-z_+}{z-z_-}\right)\right),$$

Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$

$$\left.\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\right|_{\partial\mathcal{C}}$$

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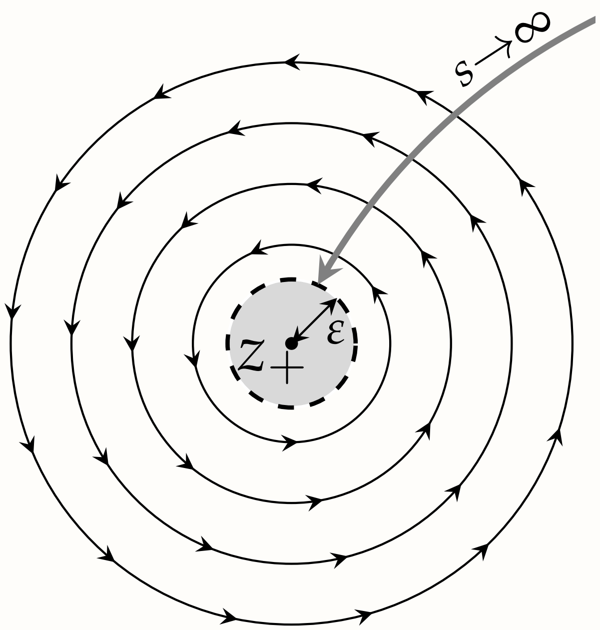

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Cut-off

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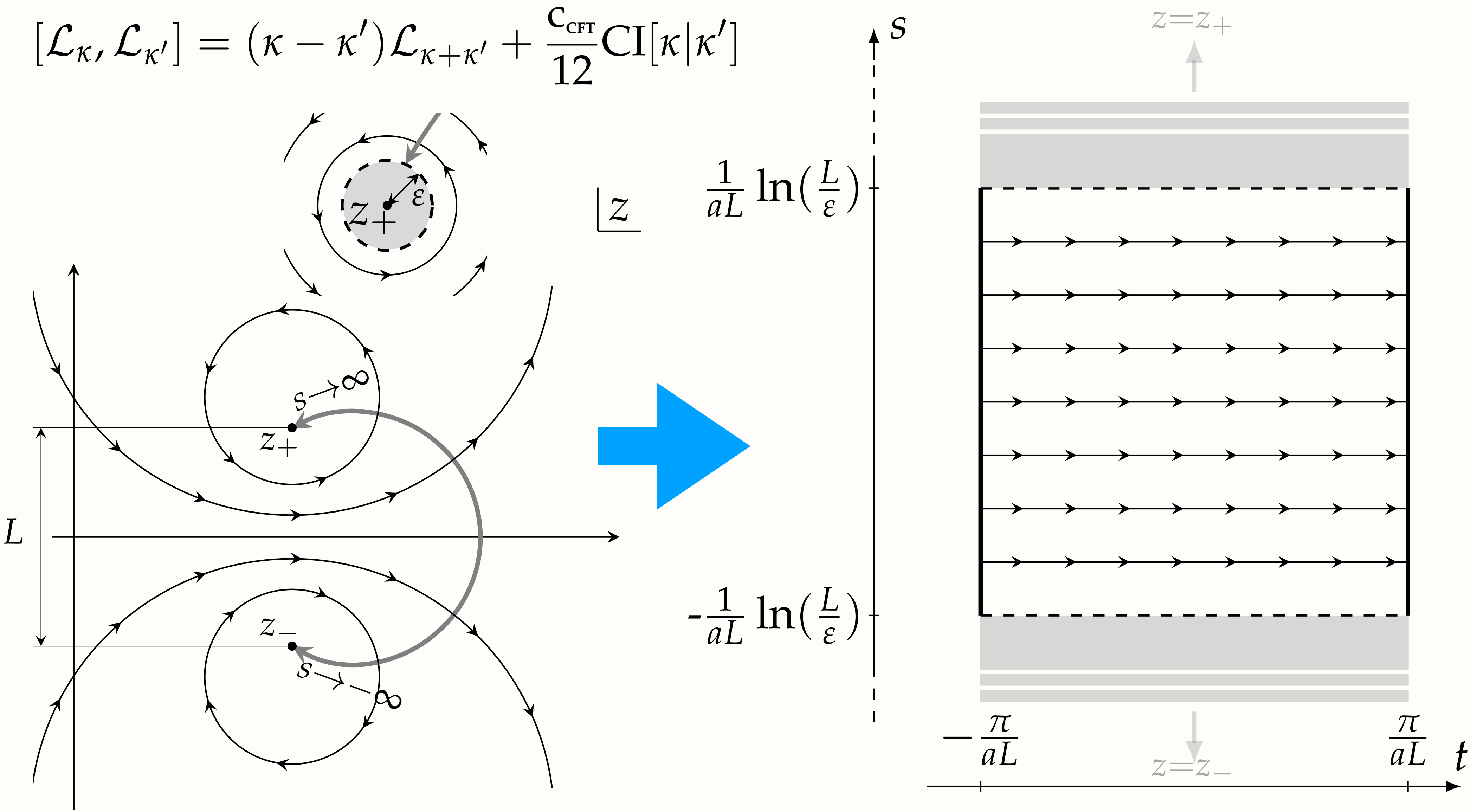
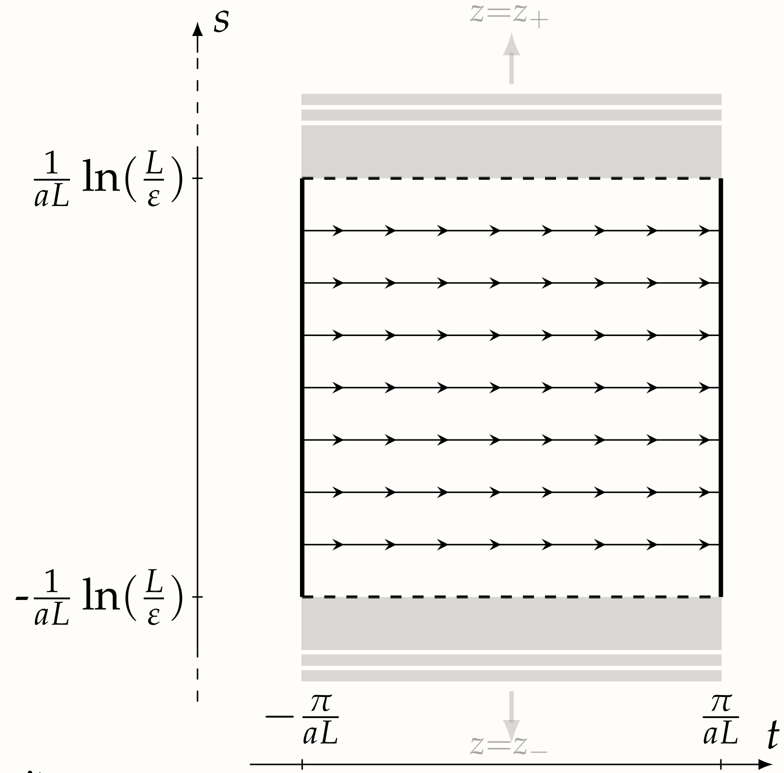


Cut-off

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$

$$\boxed{2i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right) = 2\pi i n,}$$

$$\kappa = \frac{\pi a L}{\ln(L/\varepsilon)}n, \quad n \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}$$



$$\begin{aligned}\text{CI}[\kappa|\kappa'] &= \left(-(b^2-4ac)\kappa + \kappa^3\right)\frac{e^{(\kappa+\kappa')t}}{2\pi i(\kappa+\kappa')} \\ &\quad \times \left[\exp\left(i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right)\right) - \exp\left(-i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right)\right)\right]\end{aligned}$$

Torus geometry

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\text{CI}[\kappa|\kappa']$$

$$\boxed{2i\frac{\kappa+\kappa'}{aL}\ln\left(\frac{L}{\varepsilon}\right) = 2\pi i n,}$$

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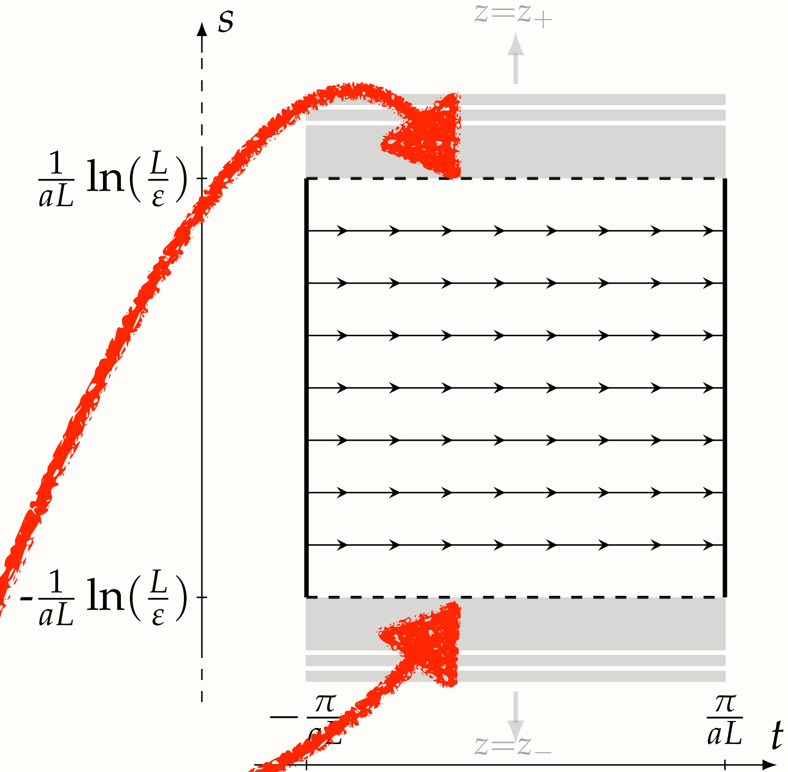


In terms of f_κ, identical

$$\text{CI}[\kappa|\kappa'] = \left(-(b^2 - 4ac)\kappa + \kappa^3\right)\frac{f_{\kappa+\kappa'}}{2\pi i(\kappa+\kappa')}\Bigg|_{\partial\mathcal{C}} = 0 \qquad \kappa + \kappa' \neq 0$$

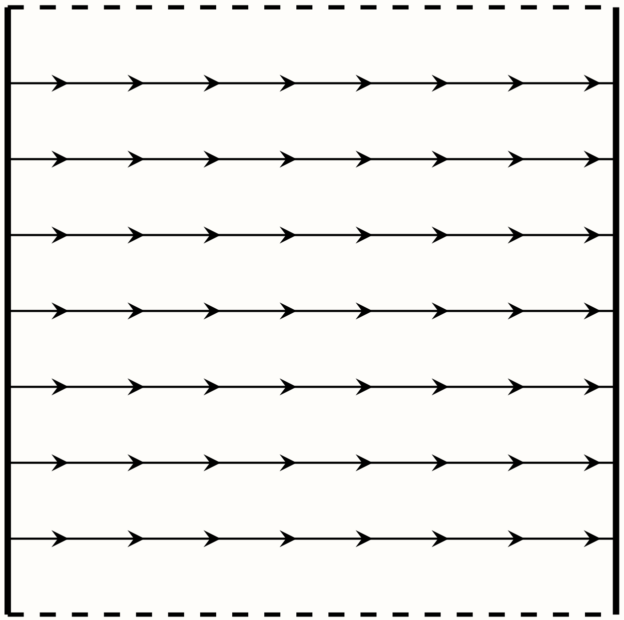
Torus geometry

After rescaling,

$$[\mathcal{L}_{\kappa}, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c_{\text{CFT}}}{12}\kappa^3\delta_{\kappa,-\kappa'}$$

Virasoro algebra on

a torus with $\tau = i\dfrac{\pi}{\ln(L/\varepsilon)}$



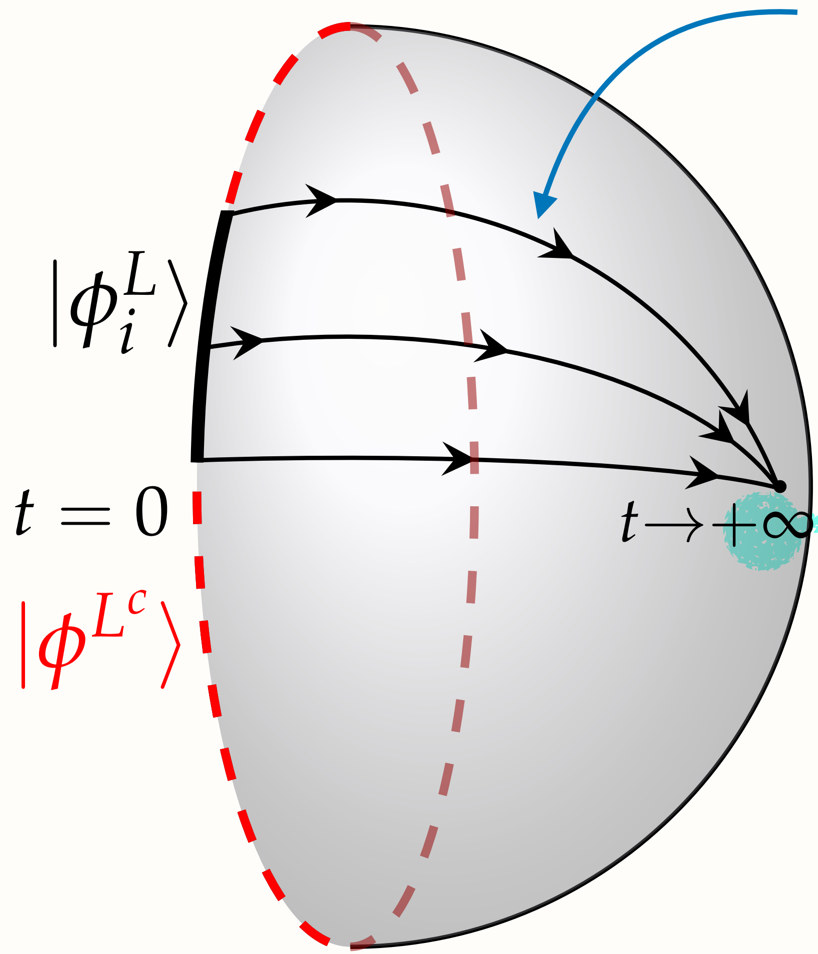
Entanglement Entropy

C. Holzhey, F. Larsen, F. Wilczek

Geometric and Renormalized Entropy in Conformal Field Theory

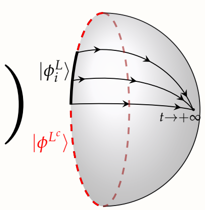
Nucl. Phys. B 424, 443 (1994)

Entanglement Entropy



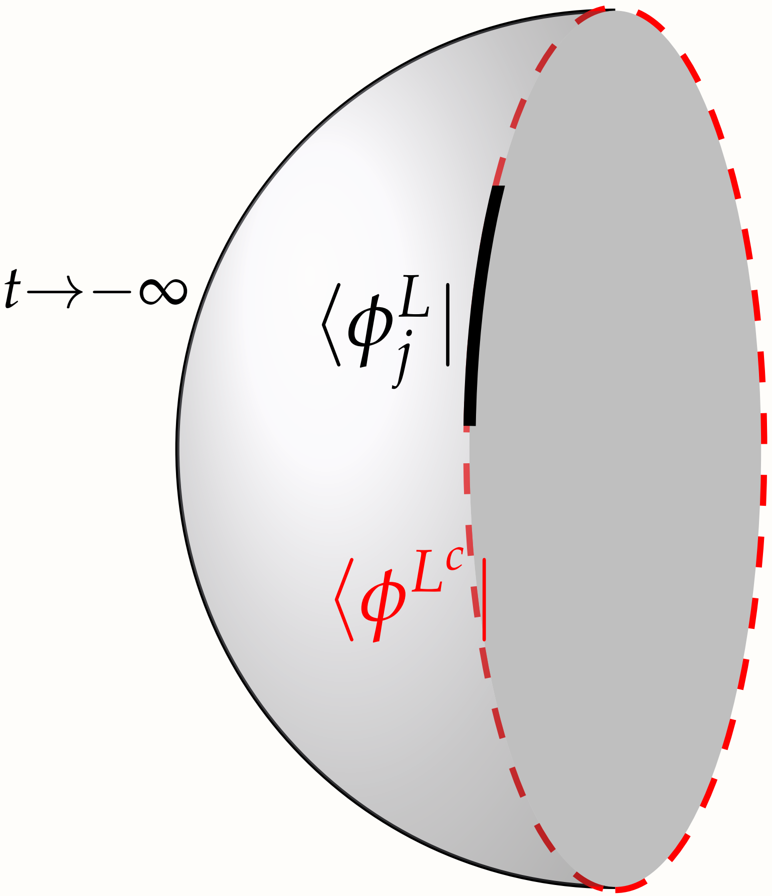
$$e^{-tH}|\phi_i^L\rangle \to |0\rangle$$

$$\langle 0|\phi_i^L\rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

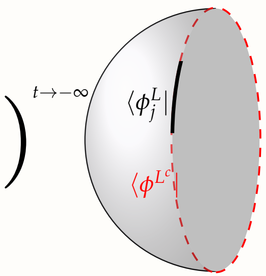


$$\int_{\phi(0,x)=\phi_i^L(x),\phi(0,x^c)=\phi^{L^c}(x^c)} \mathcal{D}\phi(t_{\geq 0},x)e^{-S}$$

Entanglement Entropy



$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

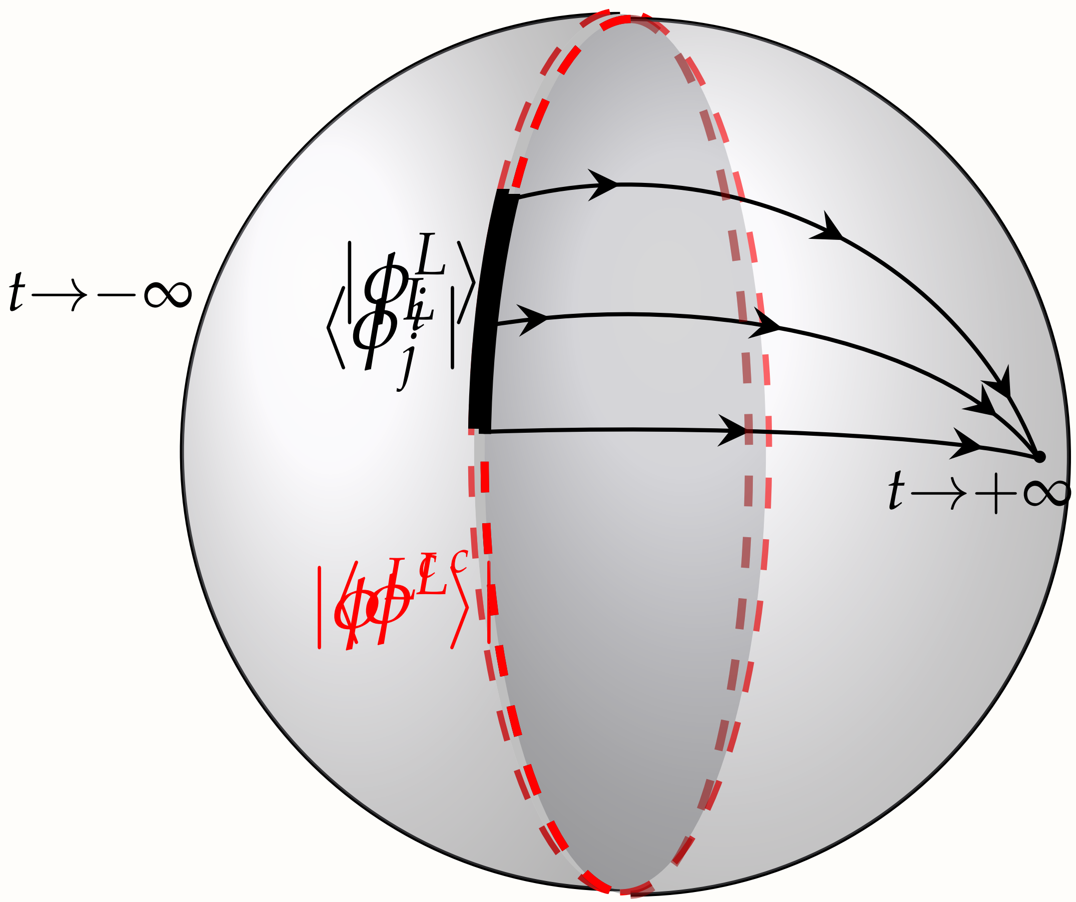


$$\int_{\phi(0,x)=\phi_j^L(x),\phi(0,x^c)=\phi^{L^c}(x^c)} \mathcal{D}\phi(t_{\leq 0}, x) e^{-S}$$

Entanglement Entropy

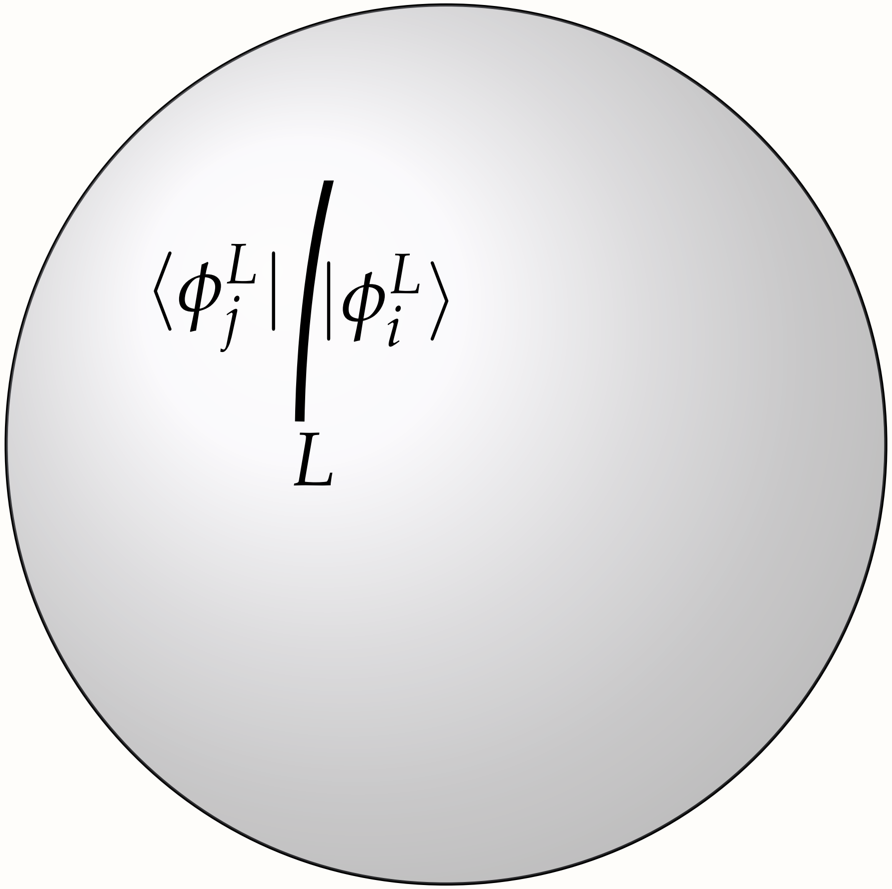
$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$

$$\langle \phi_j^L | 0 \rangle = \int \mathcal{D}\phi^{L^c}(x^c)$$



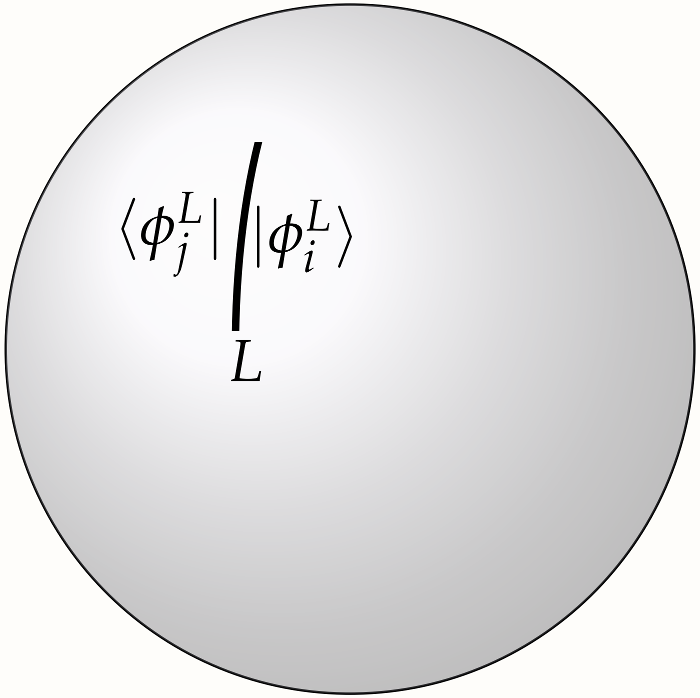
Entanglement Entropy

$$\langle \phi_j^L | 0 \rangle \langle 0 | \phi_i^L \rangle = \int_{\phi(0^-,x)=\phi_j^L(x),\phi(0^+,x)=\phi_i^L(x)} \mathcal{D}\phi(t,x) e^{-S}$$



$$= \left(\mathrm{tr}_{L^c} \left(|0\rangle\langle 0|\right)\right)_{ij}$$

Entanglement Entropy



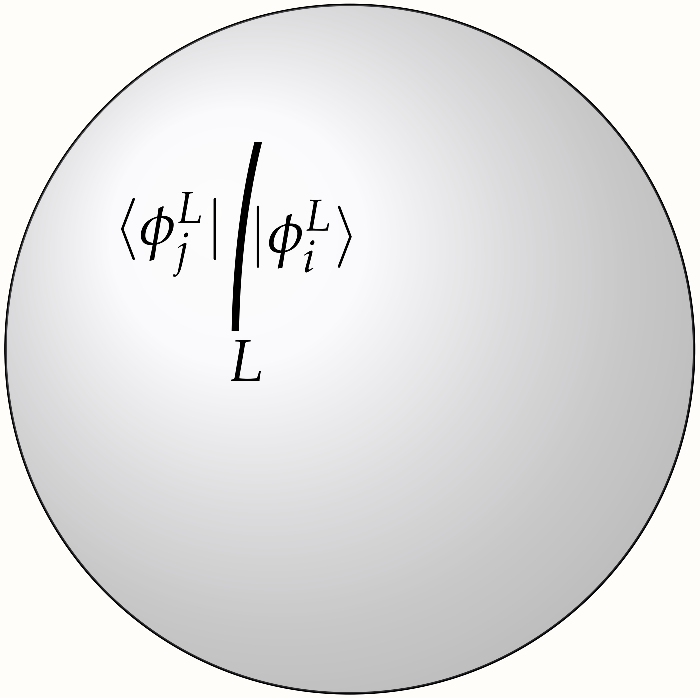
reduced density matrix

$$= \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ij} \equiv Z \times (\rho)_{ij}$$

$$Z = \sum_i \langle \phi_i^L | 0 \rangle \langle 0 | \phi_i^L \rangle = \sum_i \left(\mathrm{tr}_{L^c}\left(|0\rangle\langle 0|\right)\right)_{ii}$$

$$\sum_i (\rho)_{ii} = 1$$

Entanglement Entropy



reduced density matrix

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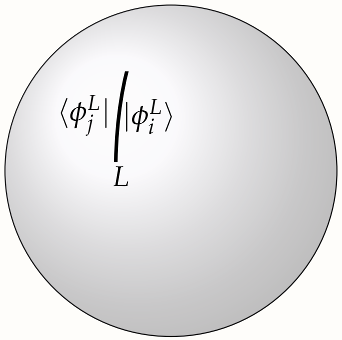
$$\sum_i (\rho)_{ii} = 1$$

$$-\mathrm{tr}\left(\rho \ln \rho \right) = S$$

Entanglement Entropy



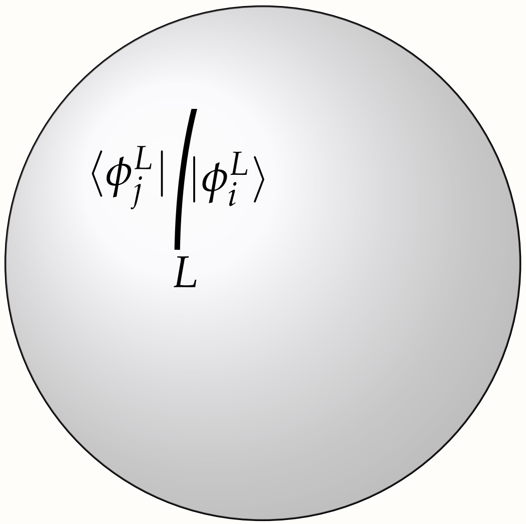
$$\equiv Z \times (\rho)_{ij}$$

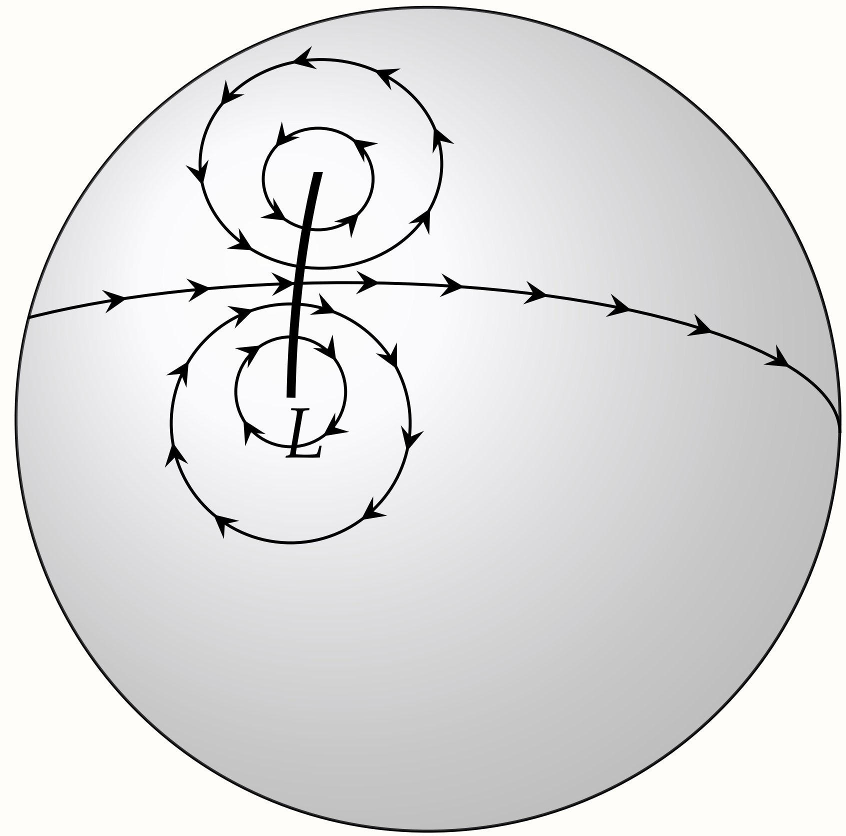
$$\rho = \frac{e^{-TH_{\mathrm{mod}}}}{\mathrm{tr}\left(e^{-TH_{\mathrm{mod}}}\right)} = \frac{e^{-TH_{\mathrm{mod}}}}{Z}$$

modular (entanglement) Hamiltonian

$$\mathrm{tr} e^{-nTH_{\mathrm{mod}}} = Z^n \mathrm{tr}\rho^n \quad \Rightarrow \quad -\frac{d}{dn}\mathrm{tr}\rho^n\Big|_{n=1} = -\mathrm{tr}\left(\rho \ln \rho\right) = S$$

Entanglement Hamiltonian



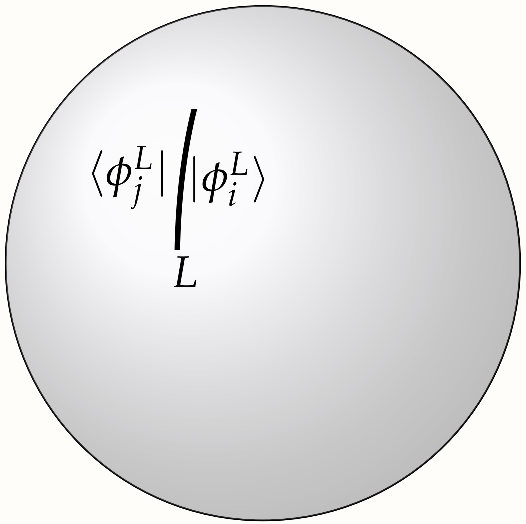


$$\equiv Z \times (\rho)_{ij}$$

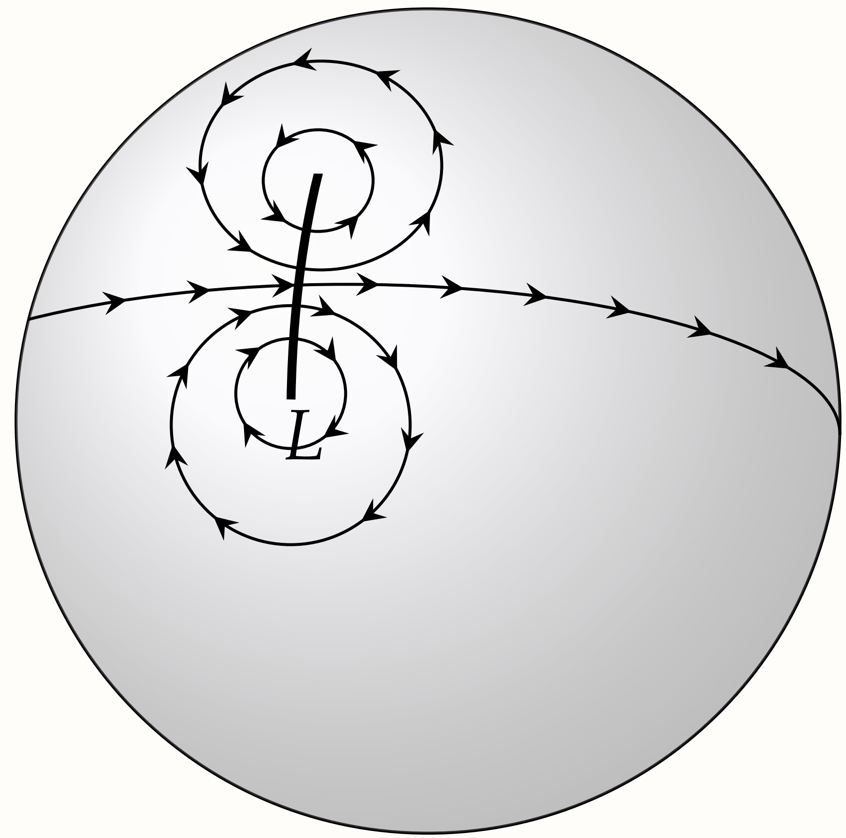
modular (entanglement) Hamiltonian

$$\rho = \frac{e^{-TH_{\mathrm{mod}}}}{\mathrm{tr}\left(e^{-TH_{\mathrm{mod}}}\right)} = \frac{e^{-TH_{\mathrm{mod}}}}{Z}$$

Entanglement Hamiltonian



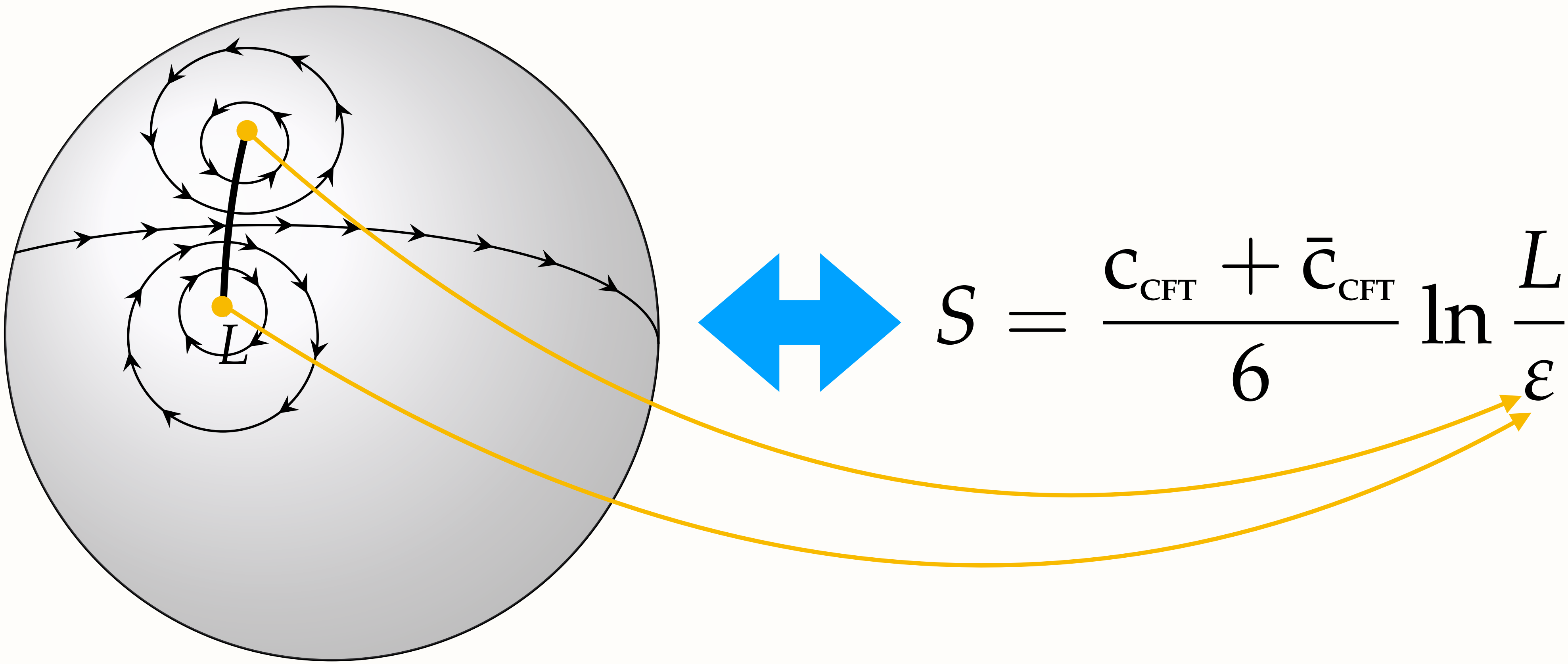
$$\equiv Z \times (\rho)_{ij}$$



modular (entanglement) Hamiltonian

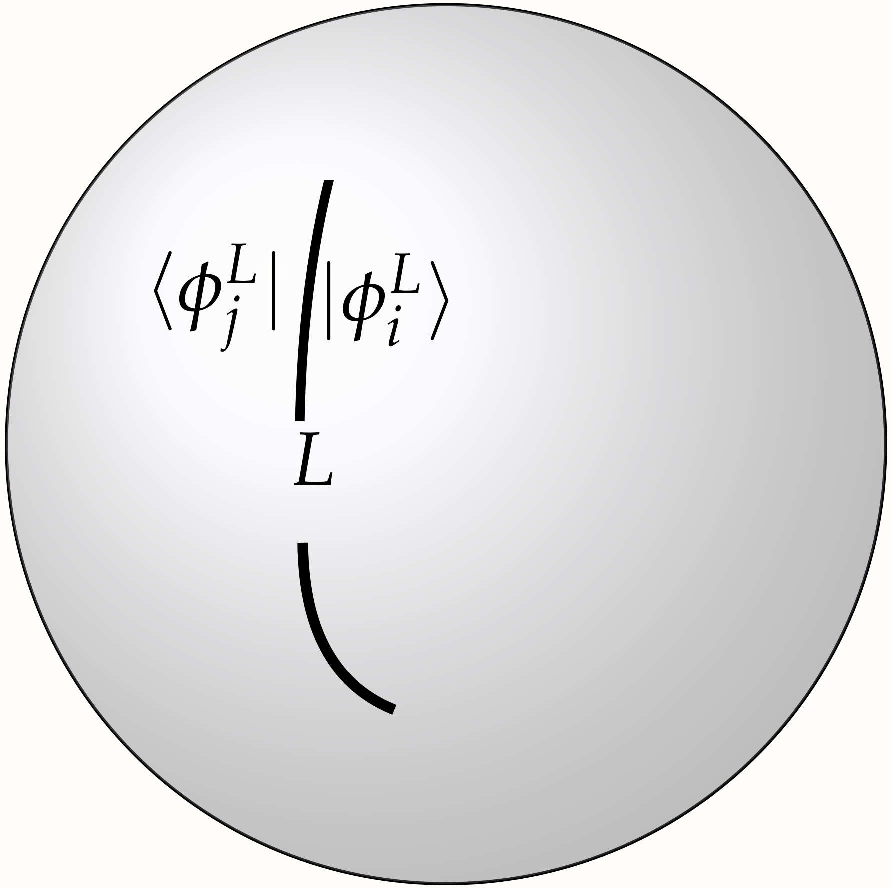
$$H_{\mathrm{mod}} = aL_1 + bL_0 + cL_{-1} + a\bar{L}_1 + b\bar{L}_0 + c\bar{L}_{-1} = \mathcal{L}_0 + \bar{\mathcal{L}}_0$$

Entanglement Hamiltonian



$$S = \frac{c_{\mathrm{CFT}} + \bar{c}_{\mathrm{CFT}}}{6} \ln \frac{L}{\varepsilon}$$

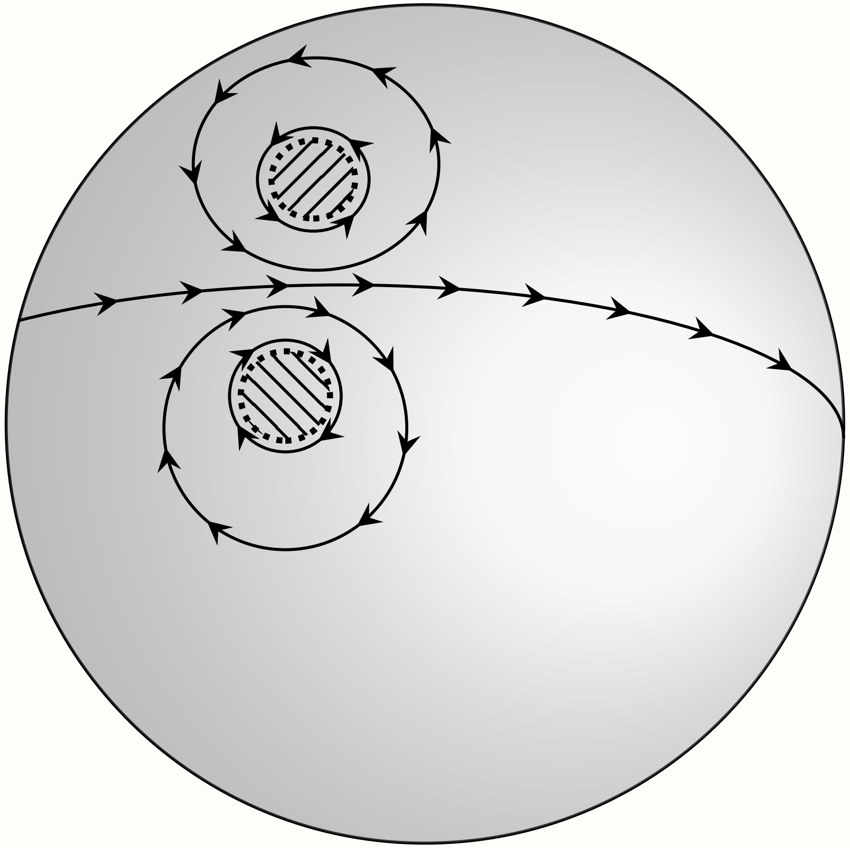
Two or more sections



J. Cardy, E. Tonni, J. Stat. Mech. (2016) 123103

G. Wong, JHEP04(2019)045

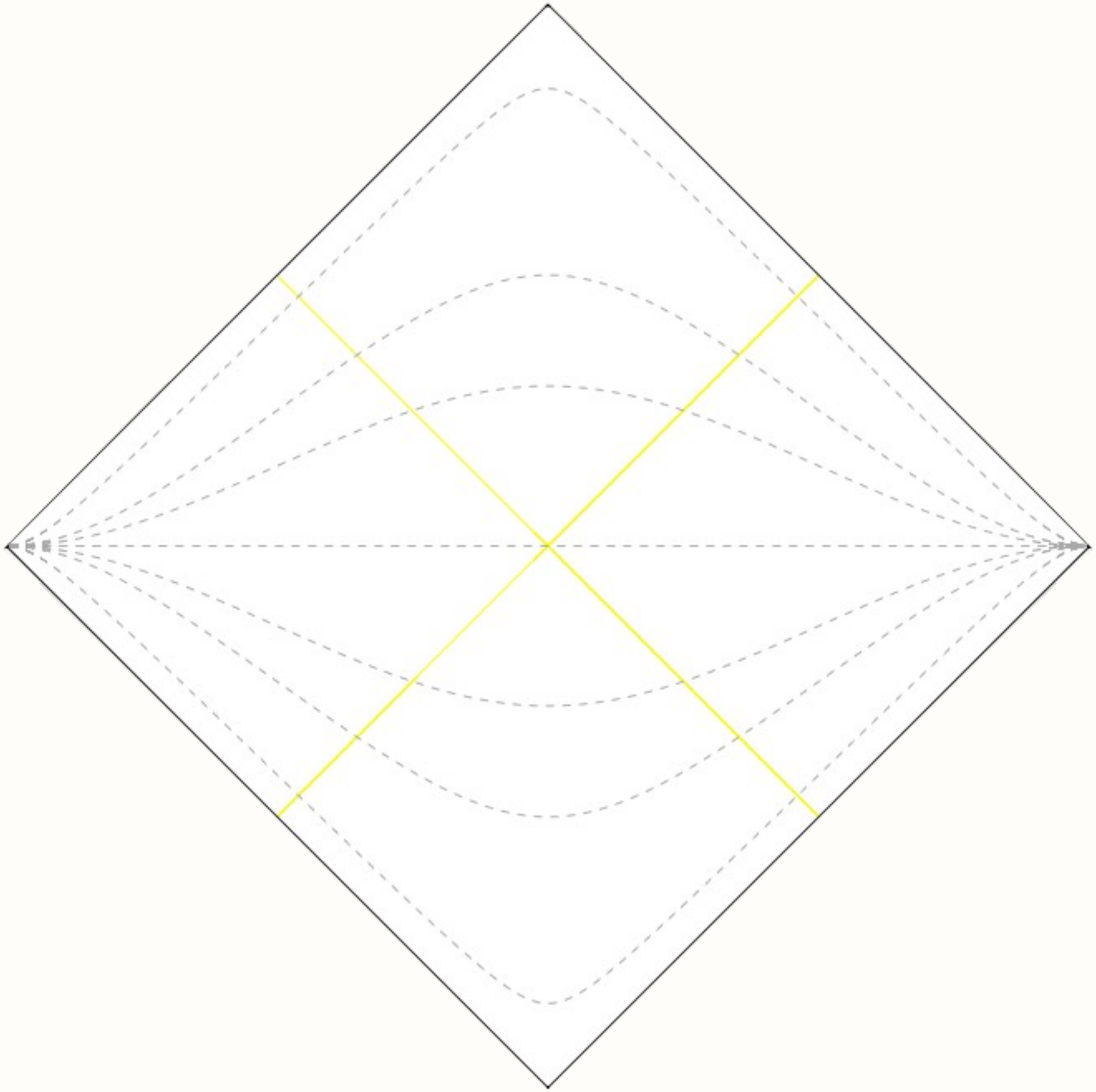
One-loop and higher



larger holes

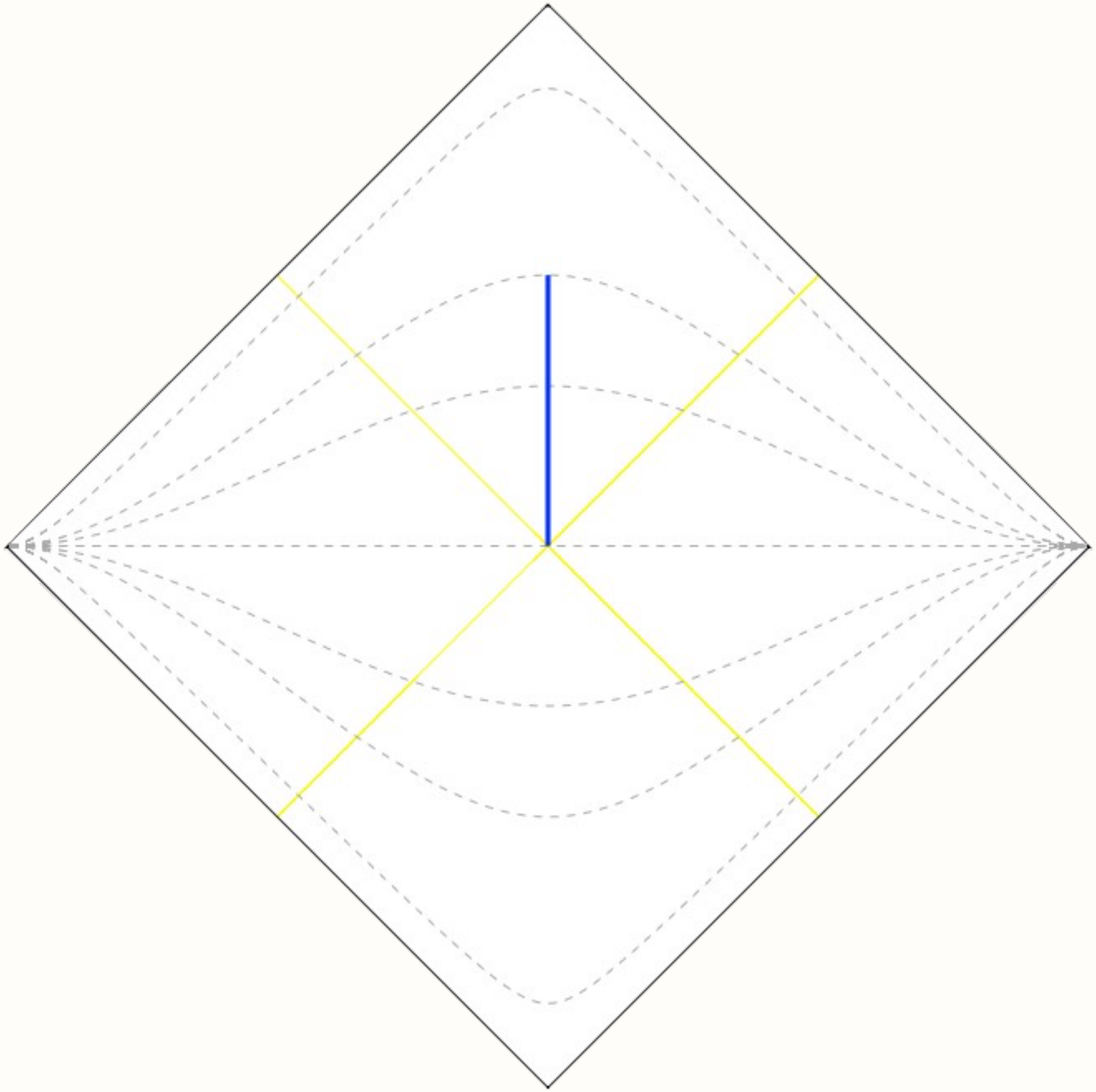
Minkowski CFT

Lüscher, Mack ('75)



Minkowski CFT

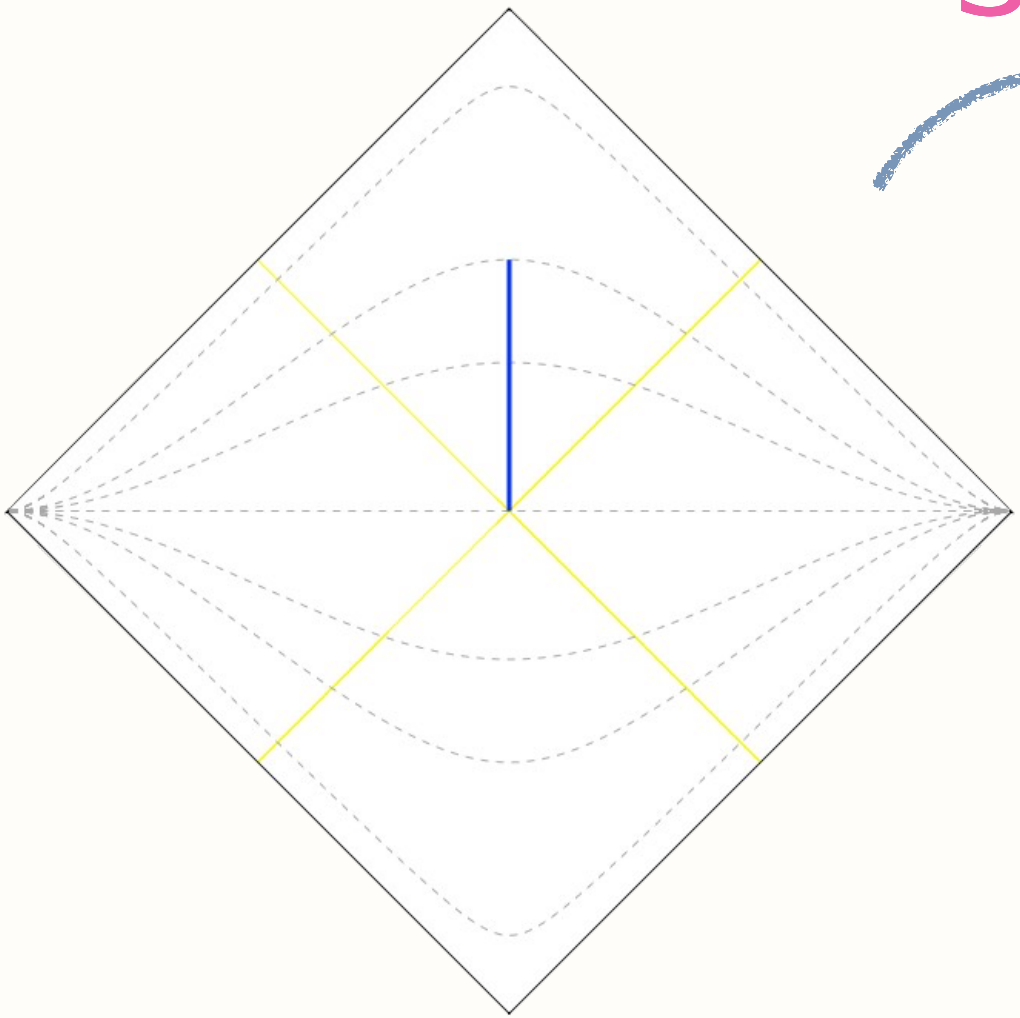
Lüscher, Mack ('75)

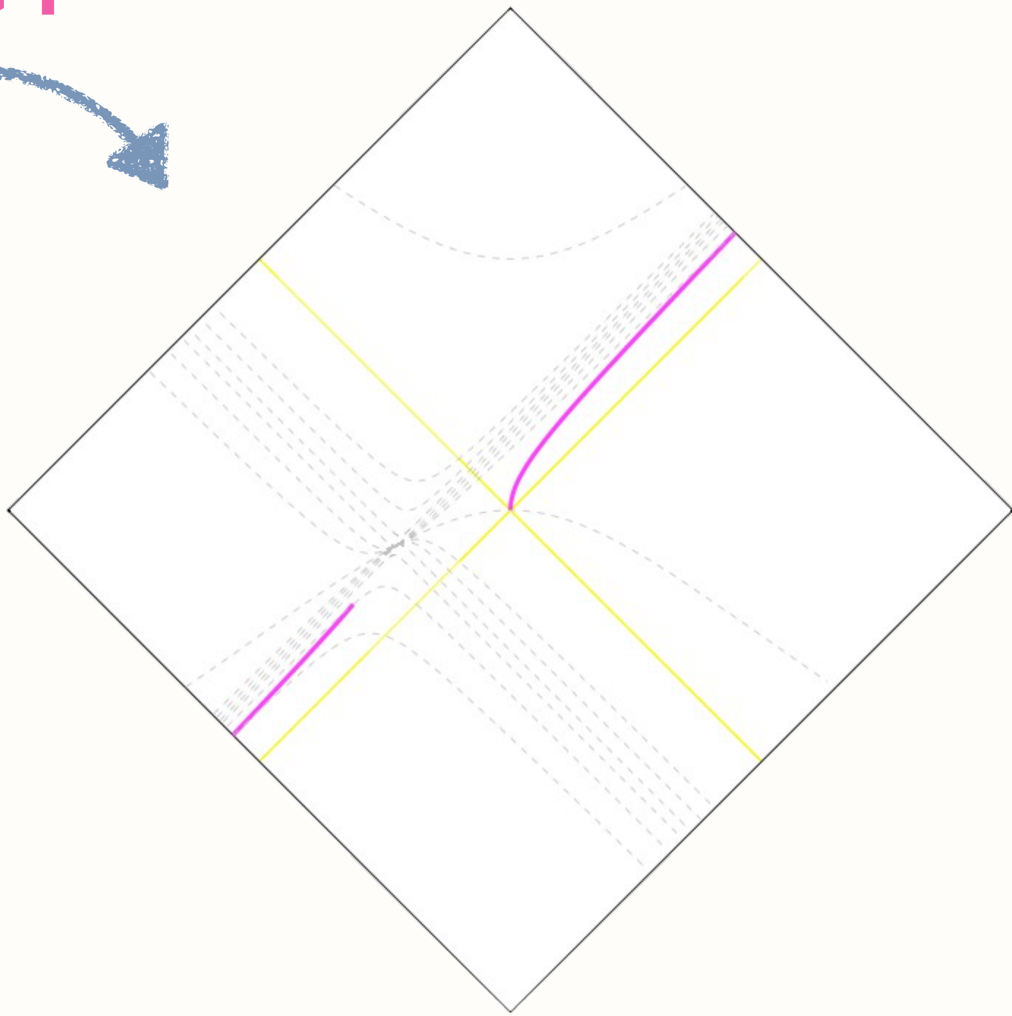


Minkowski CFT

Lüscher, Mack ('75)

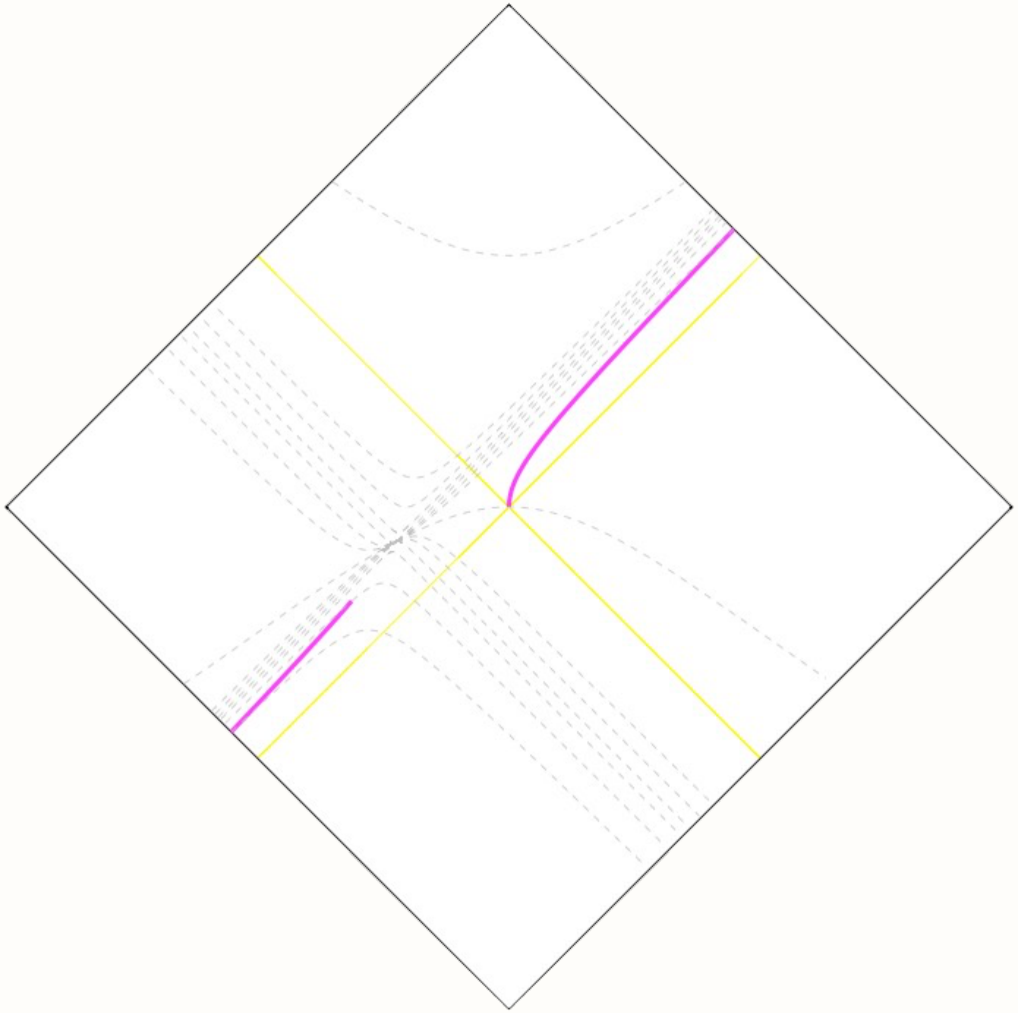
SCT





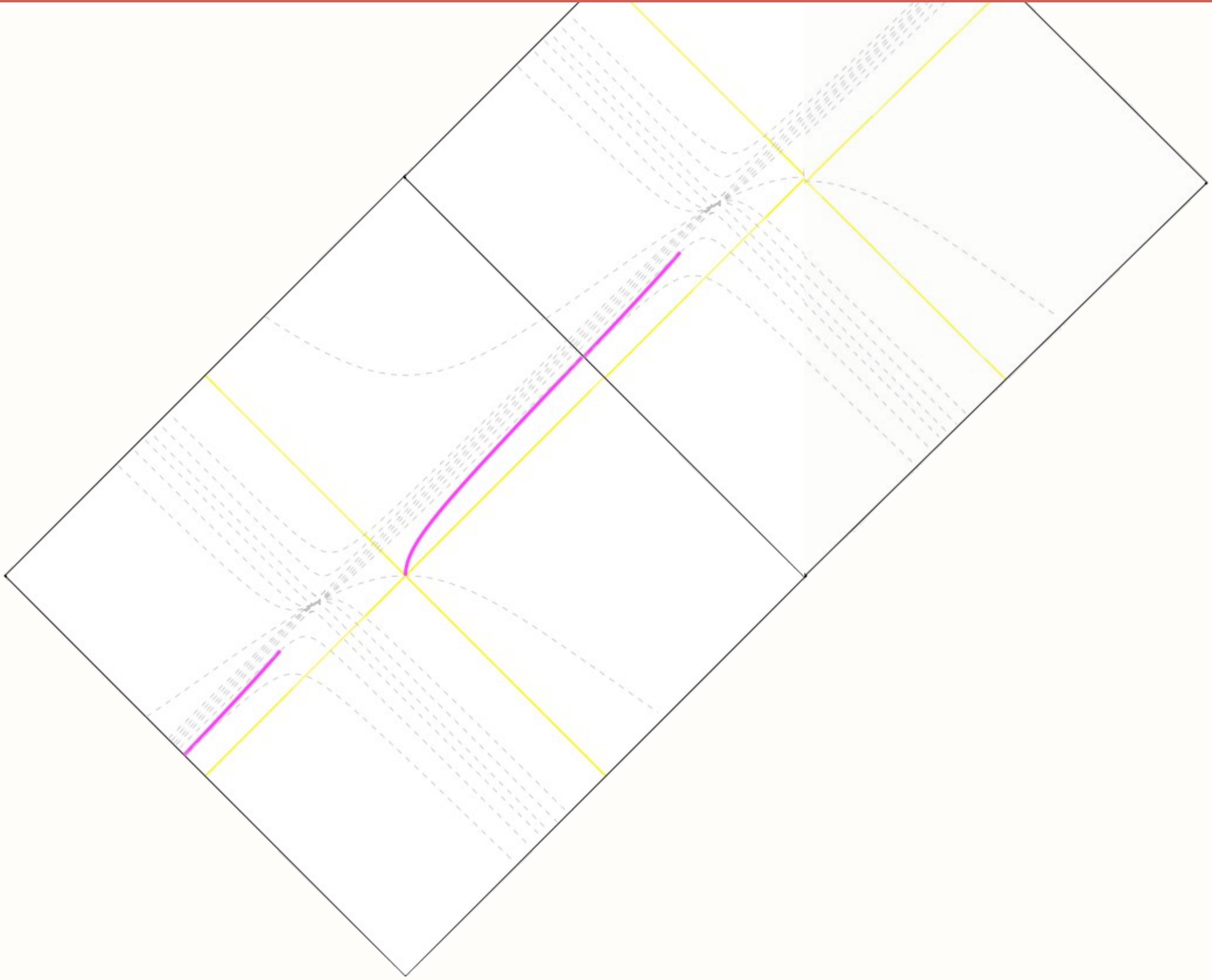
Minkowski CFT

Lüscher, Mack ('75)



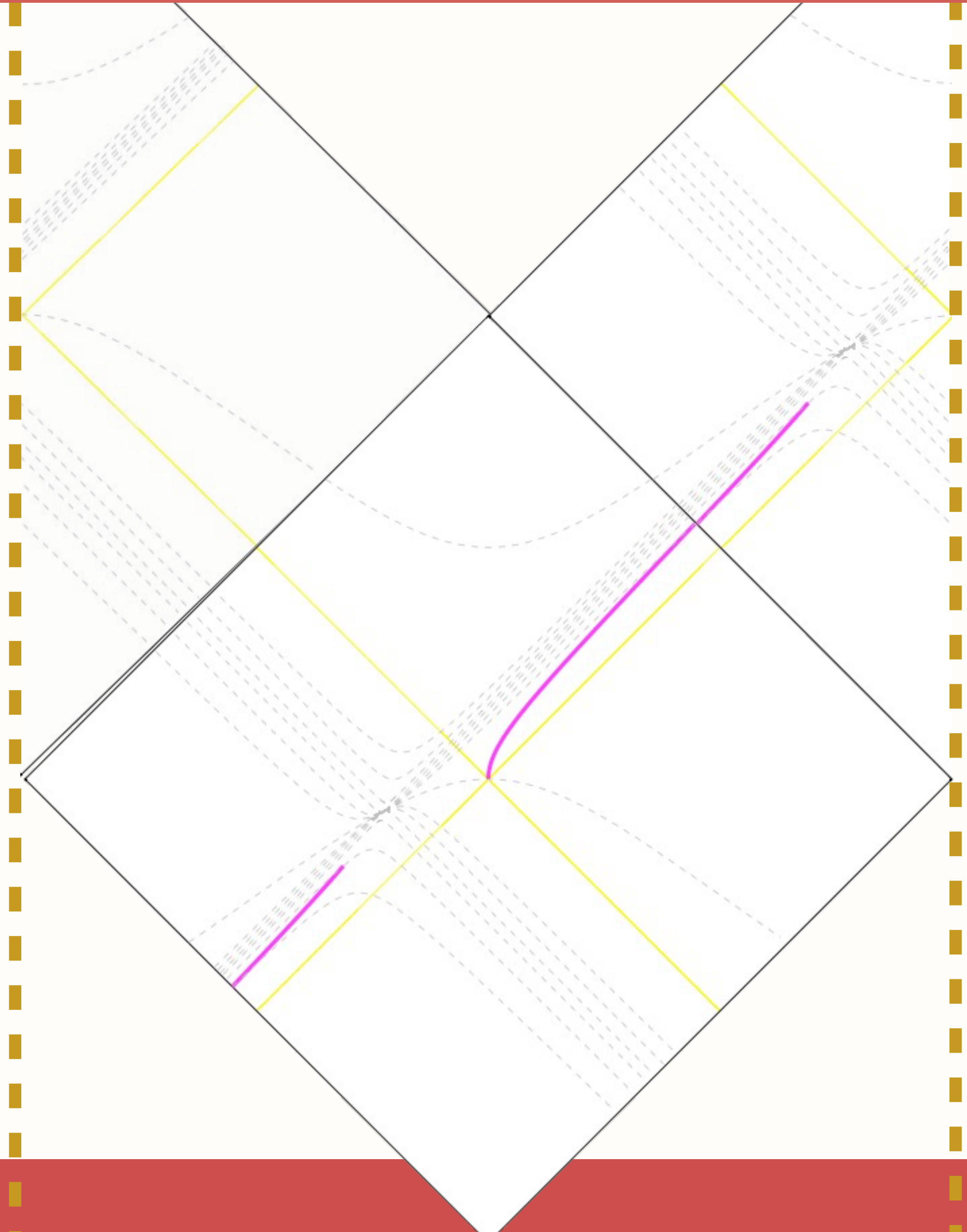
Minkowski CFT

Lüscher, Mack ('75)



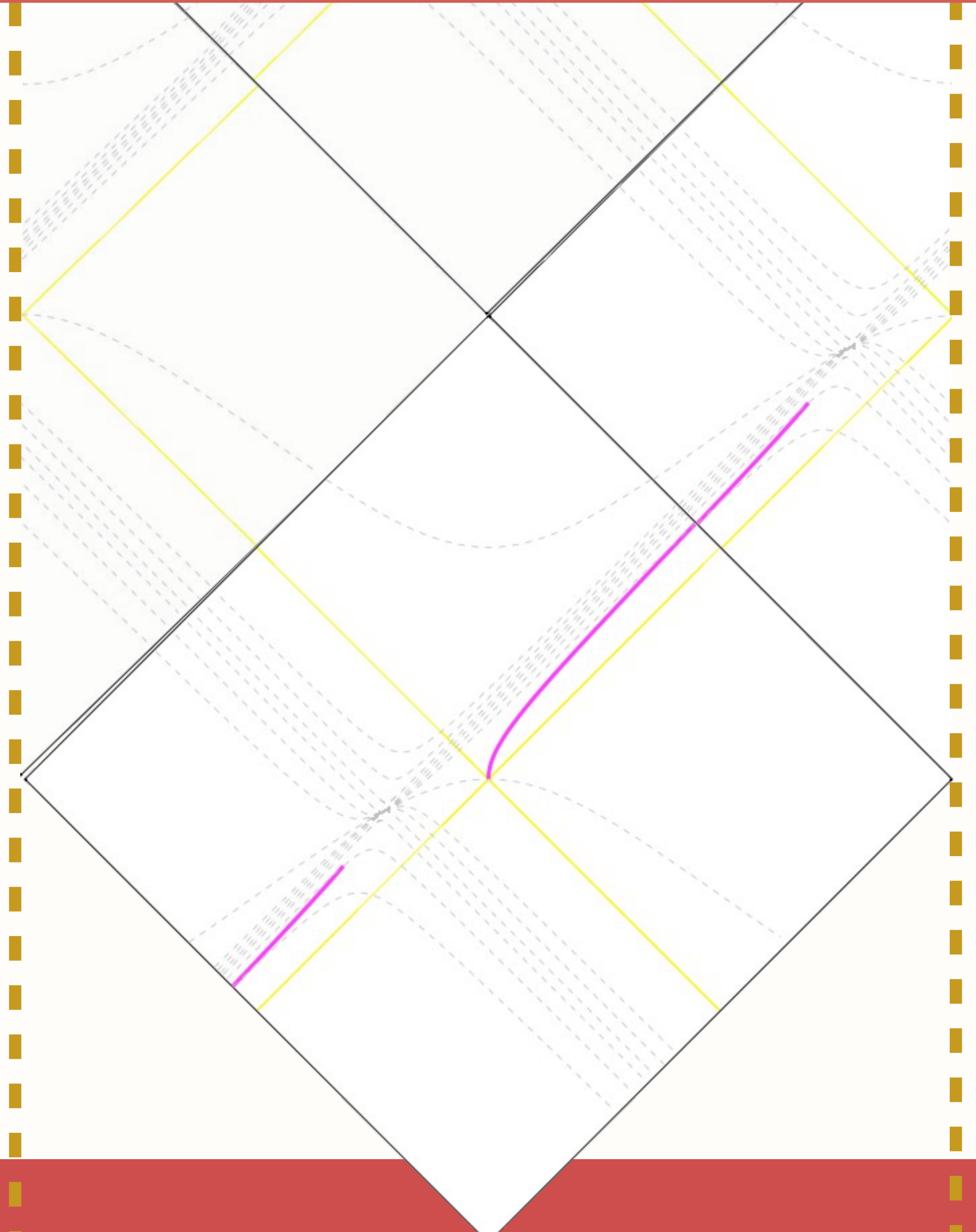
Minkowski CFT

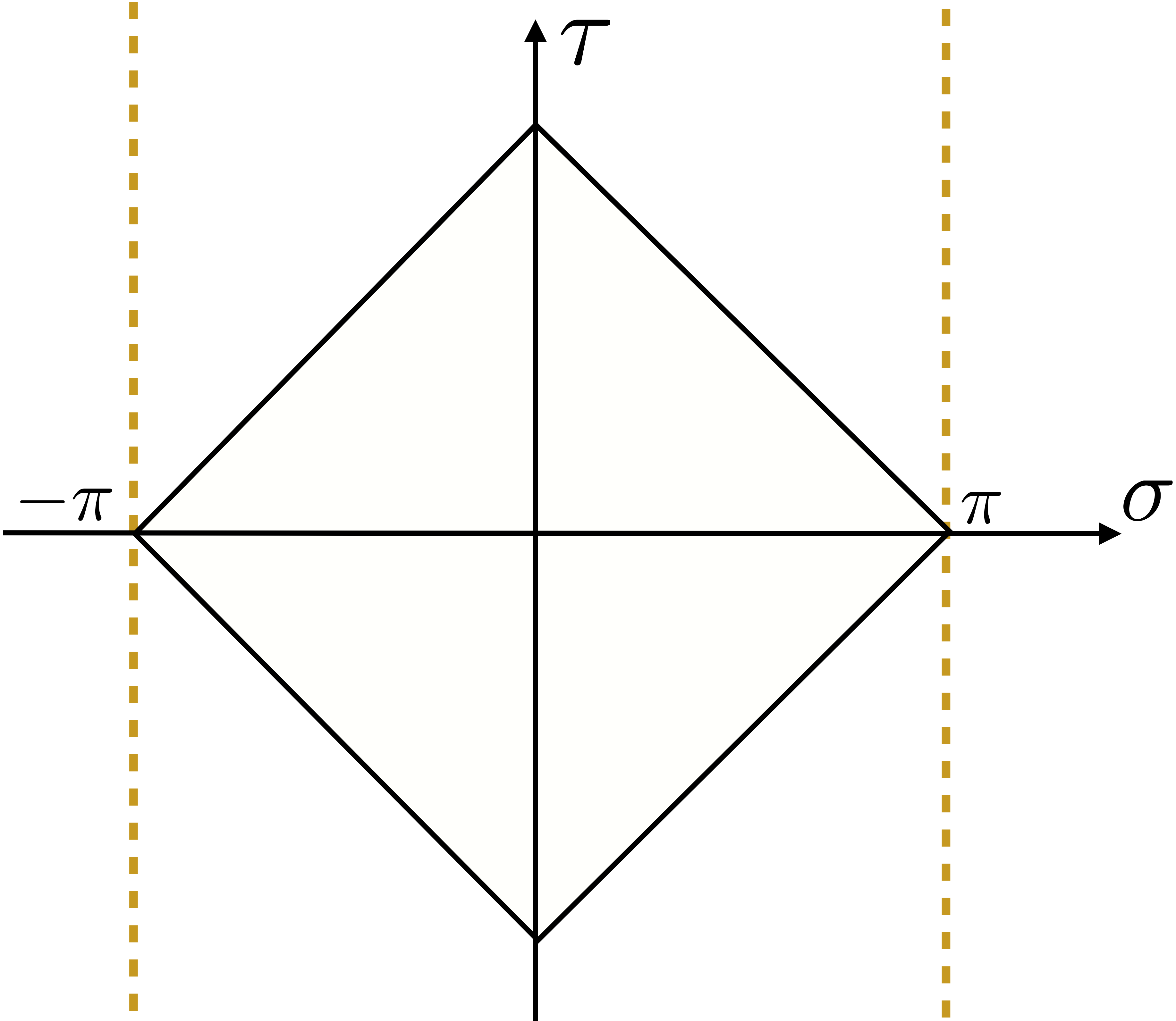
Lüscher, Mack ('75)

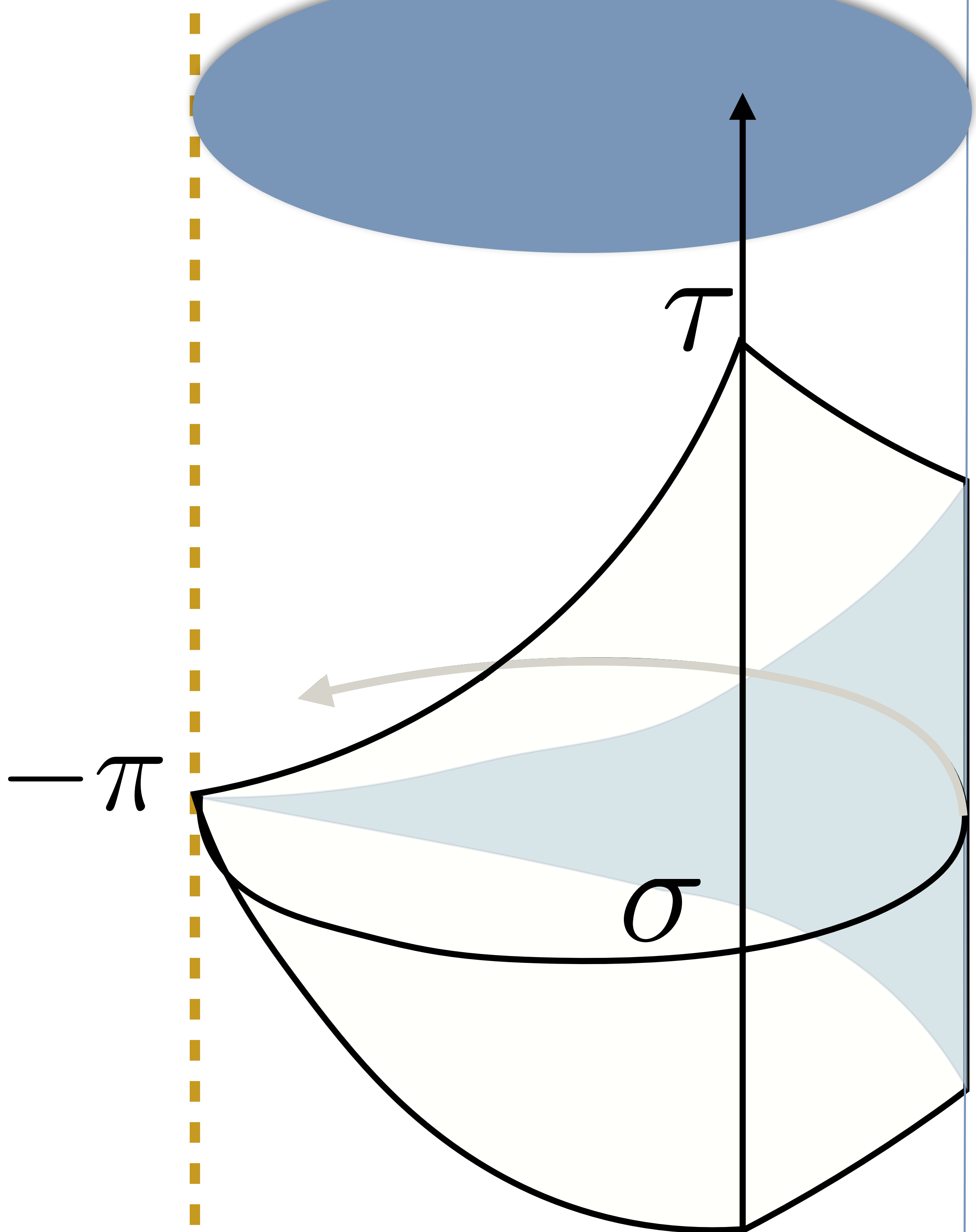


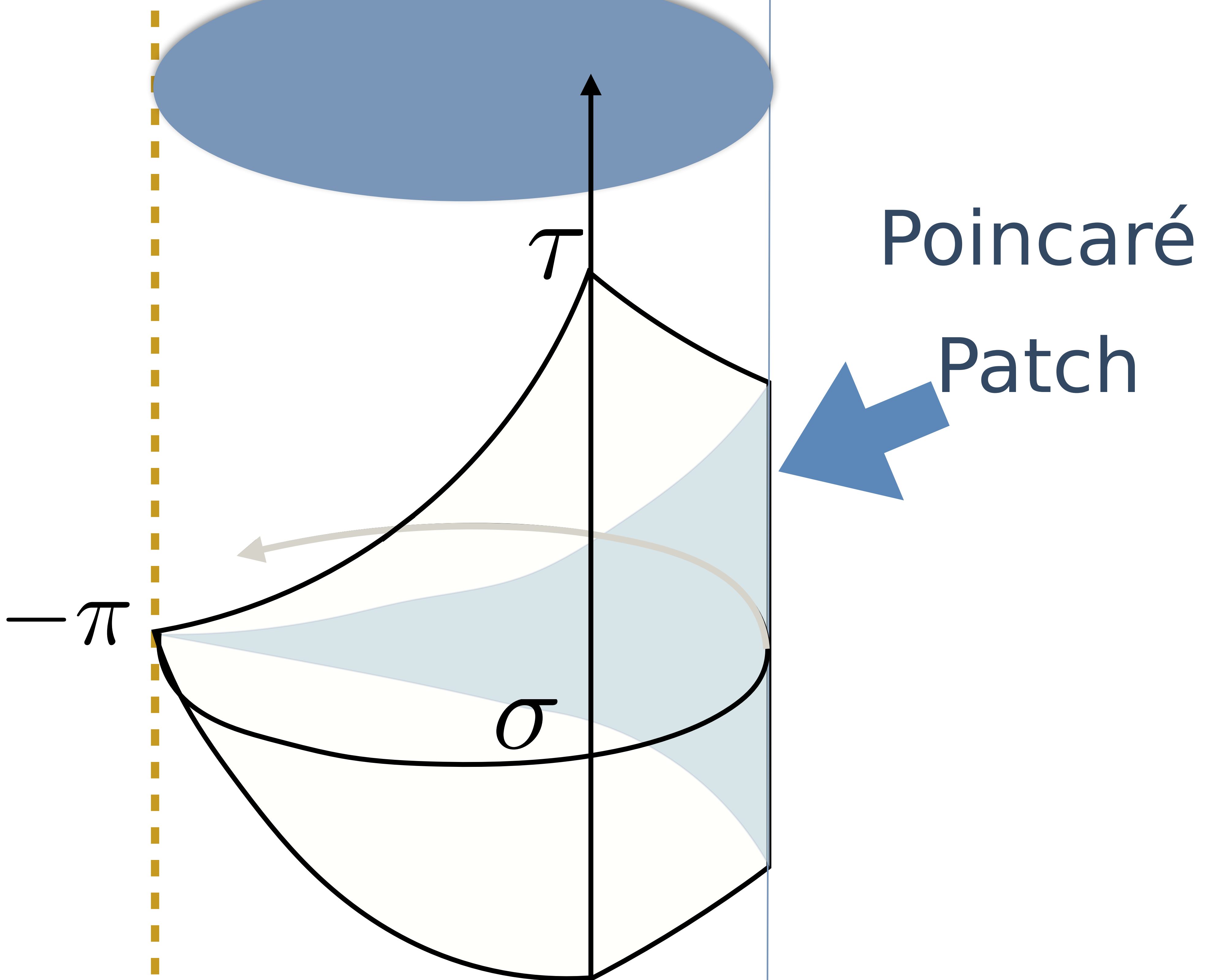
Minkowski CFT

Lüscher, Mack ('75)

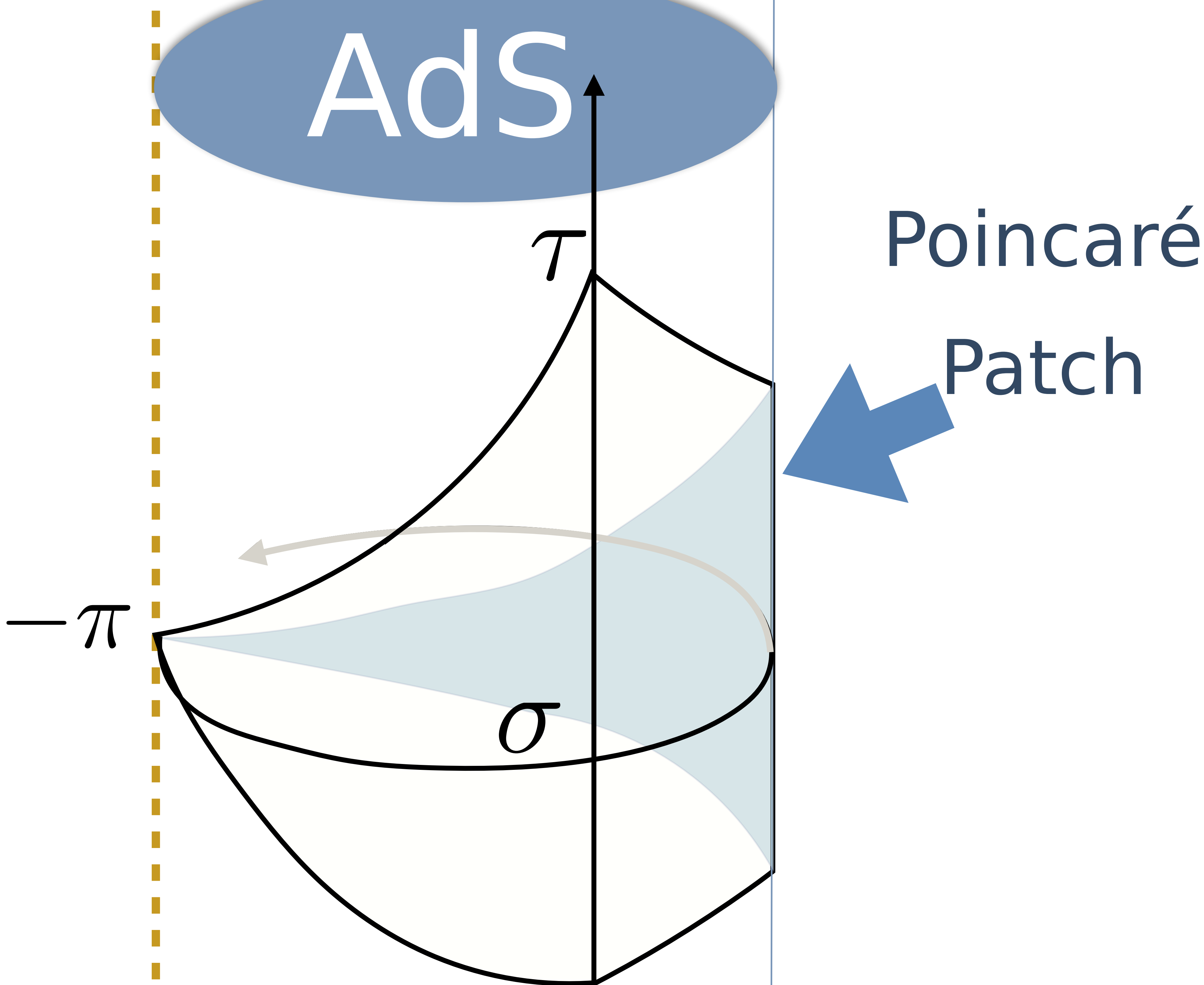


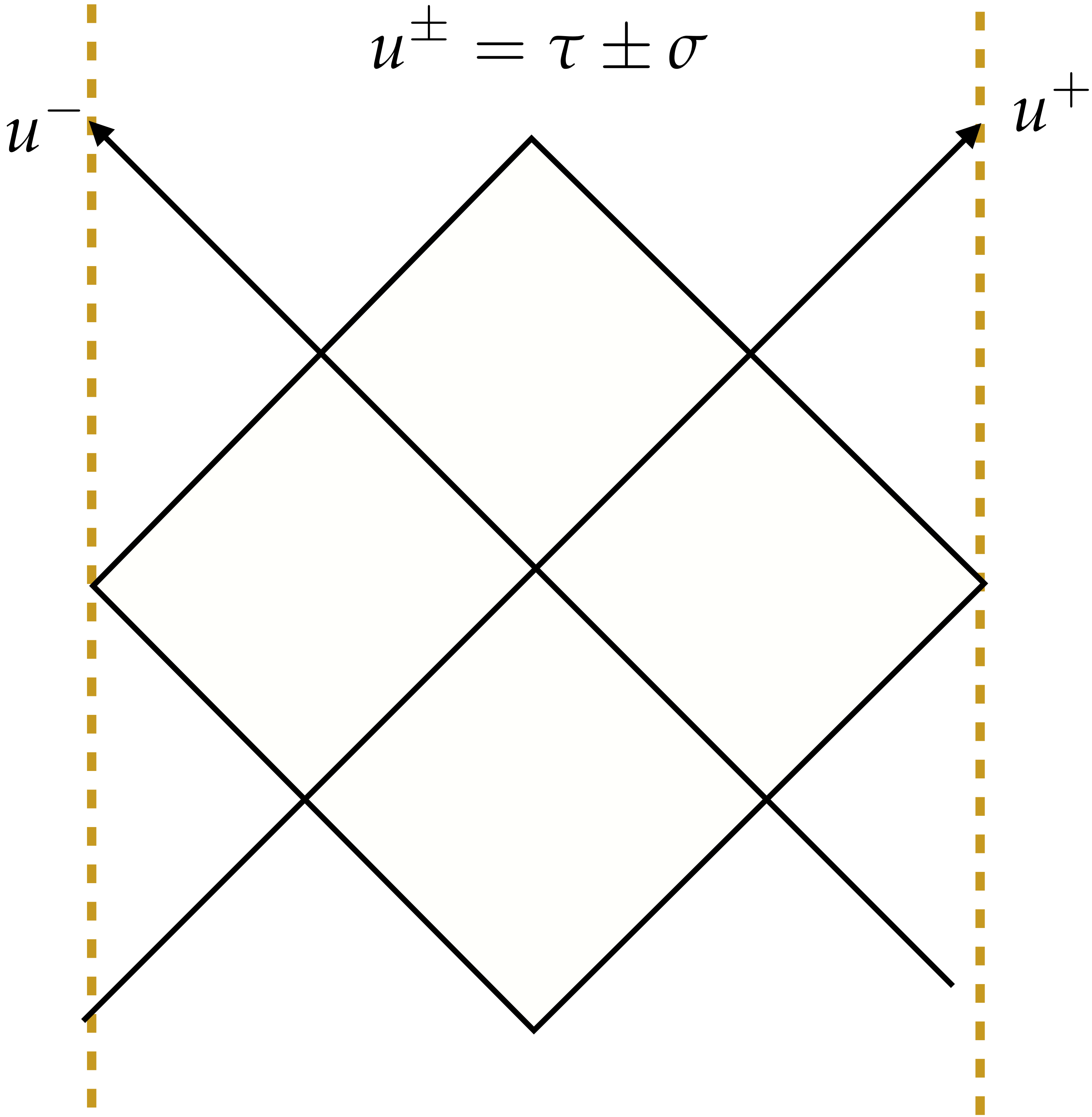

τ
σ
$-\pi$
π


τ
$-\pi$
σ



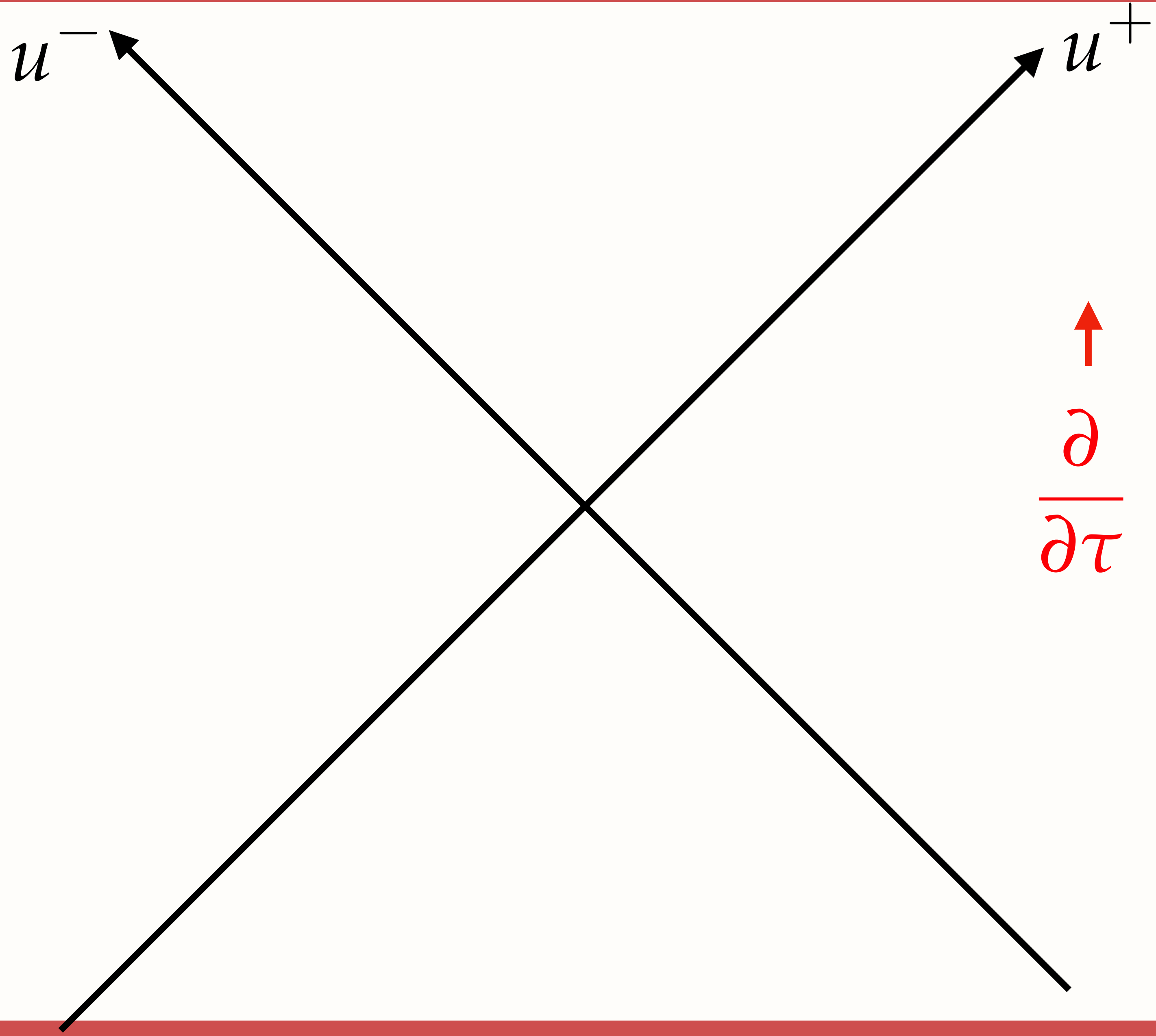
τ
$-\pi$
σ
Poincaré
Patch


AdS
τ
Poincaré
Patch
$-\pi$
σ



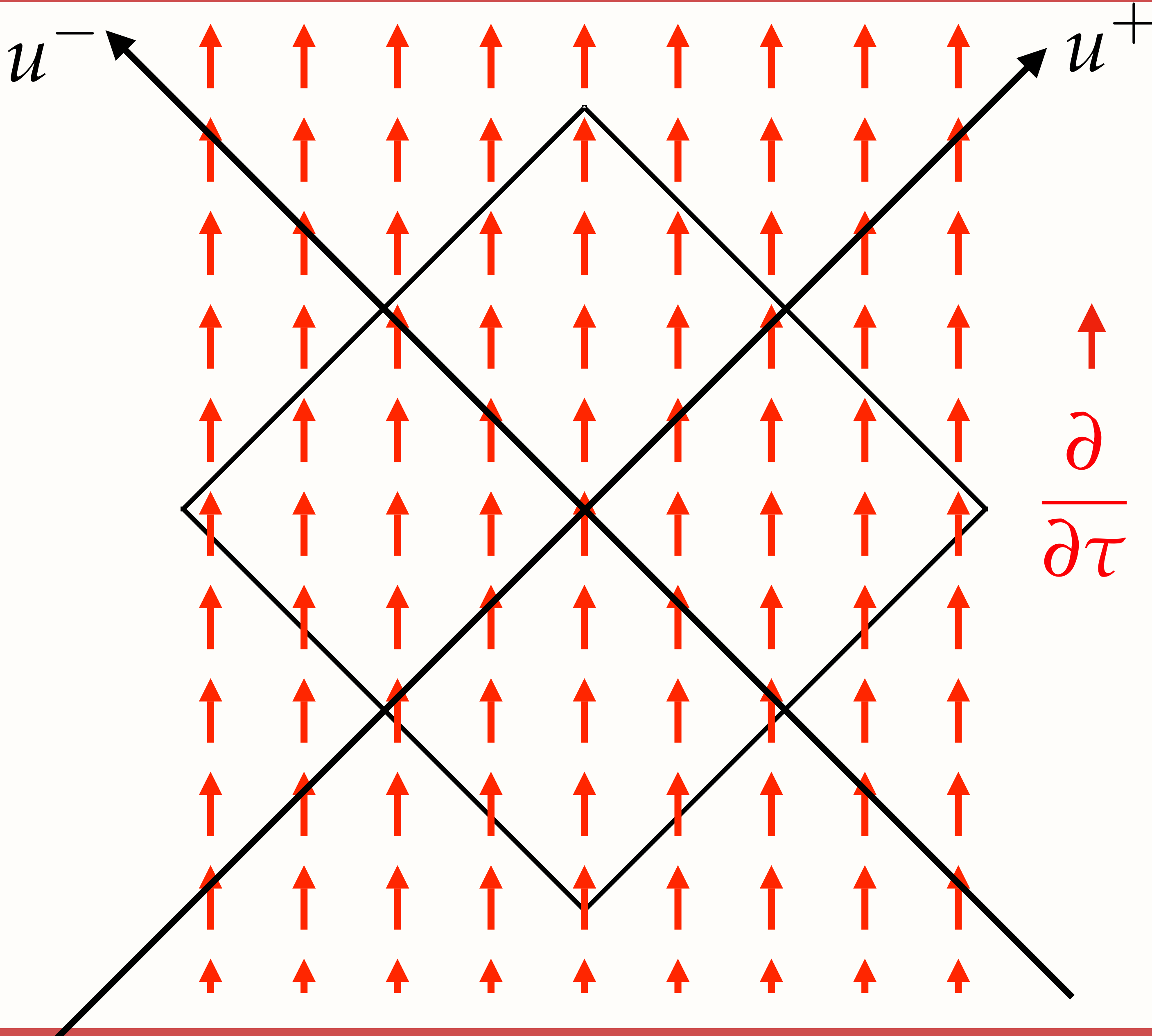
$u^{\pm} = \tau \pm \sigma$
u^{-}
u^{+}

Minkowski CFT



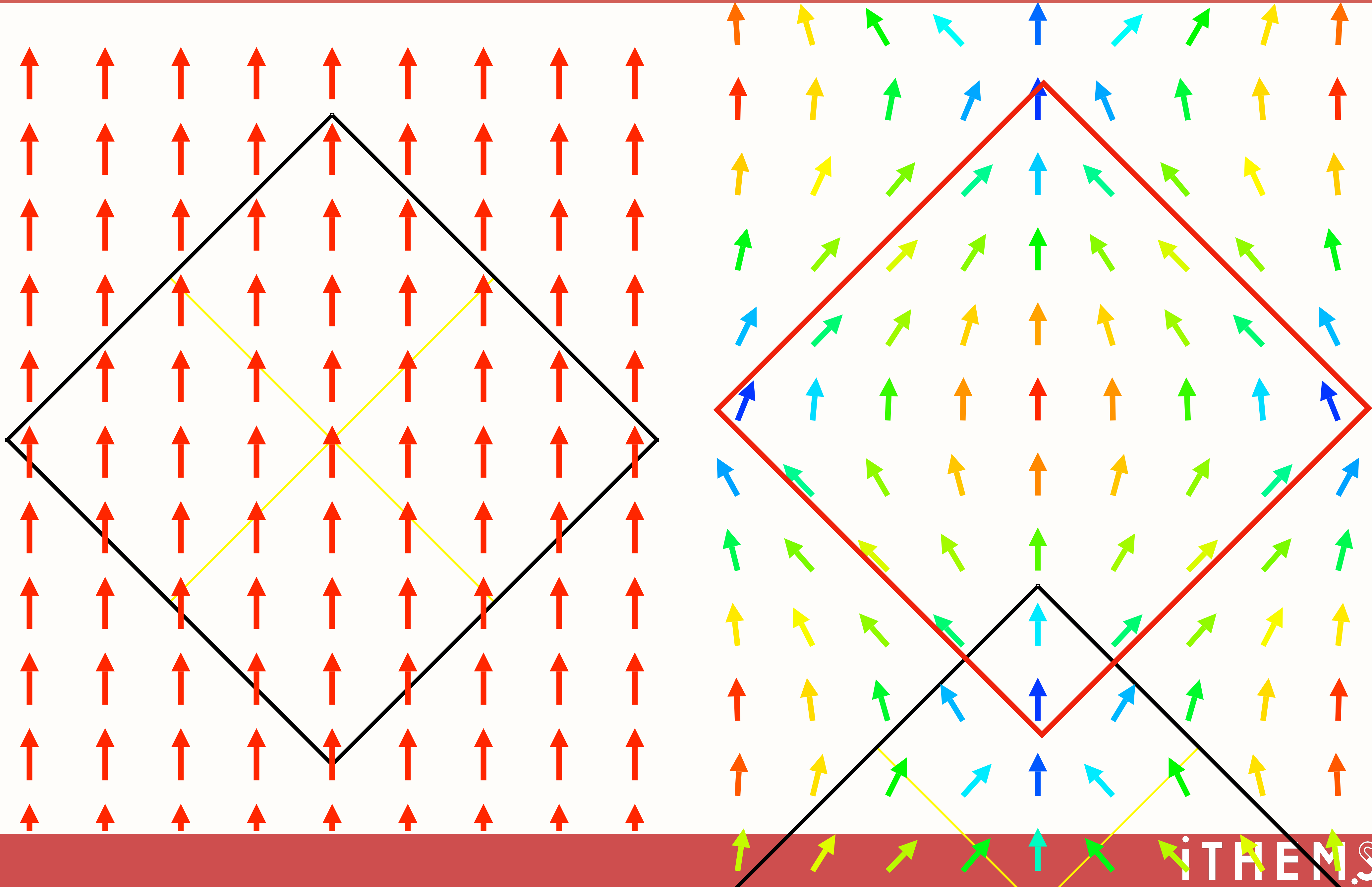
$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}$$

Minkowski CFT

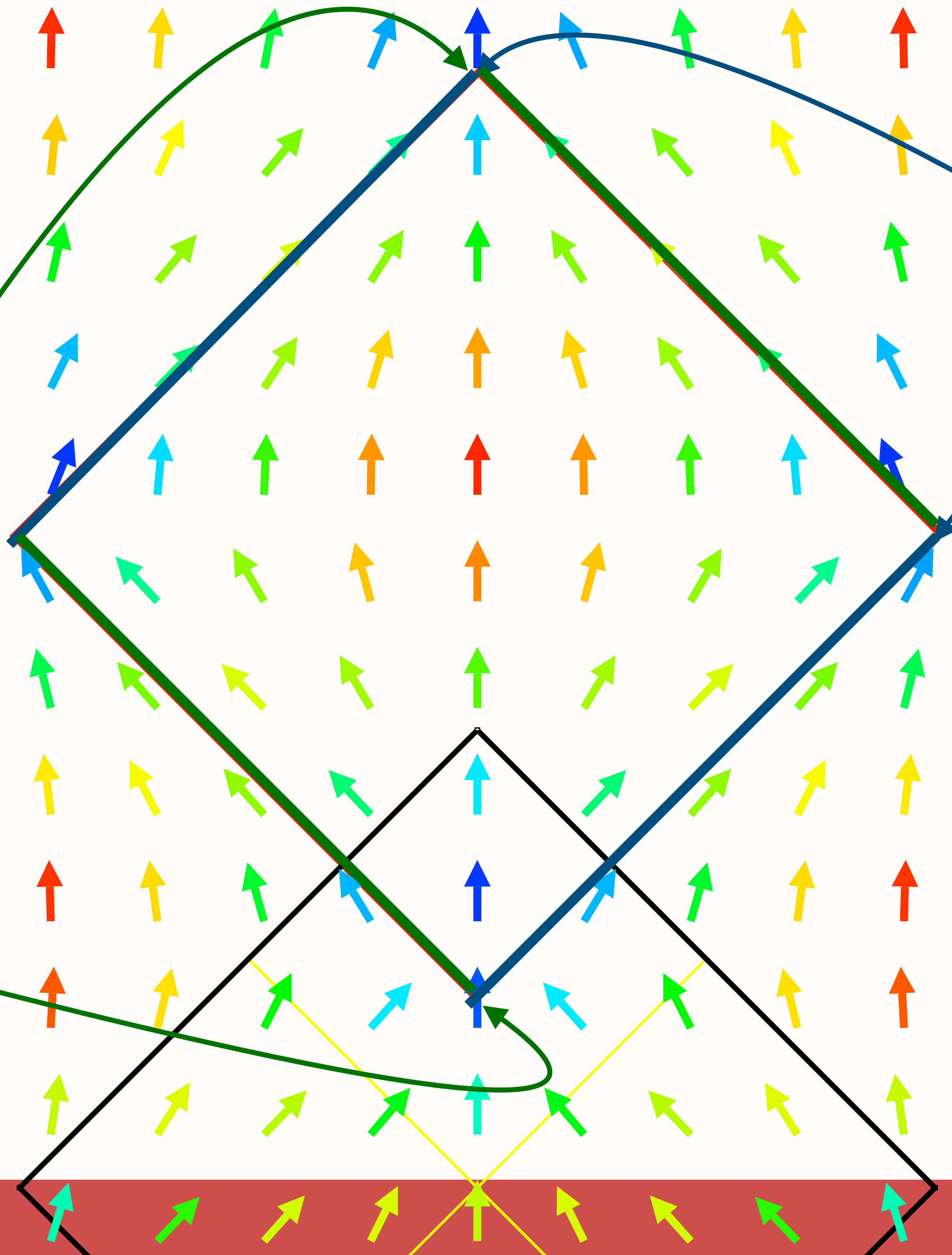


$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}$$

Minkowski CFT



Minkowski CFT



zero normal velocity

zero normal velocity

time flow confined in a diamond

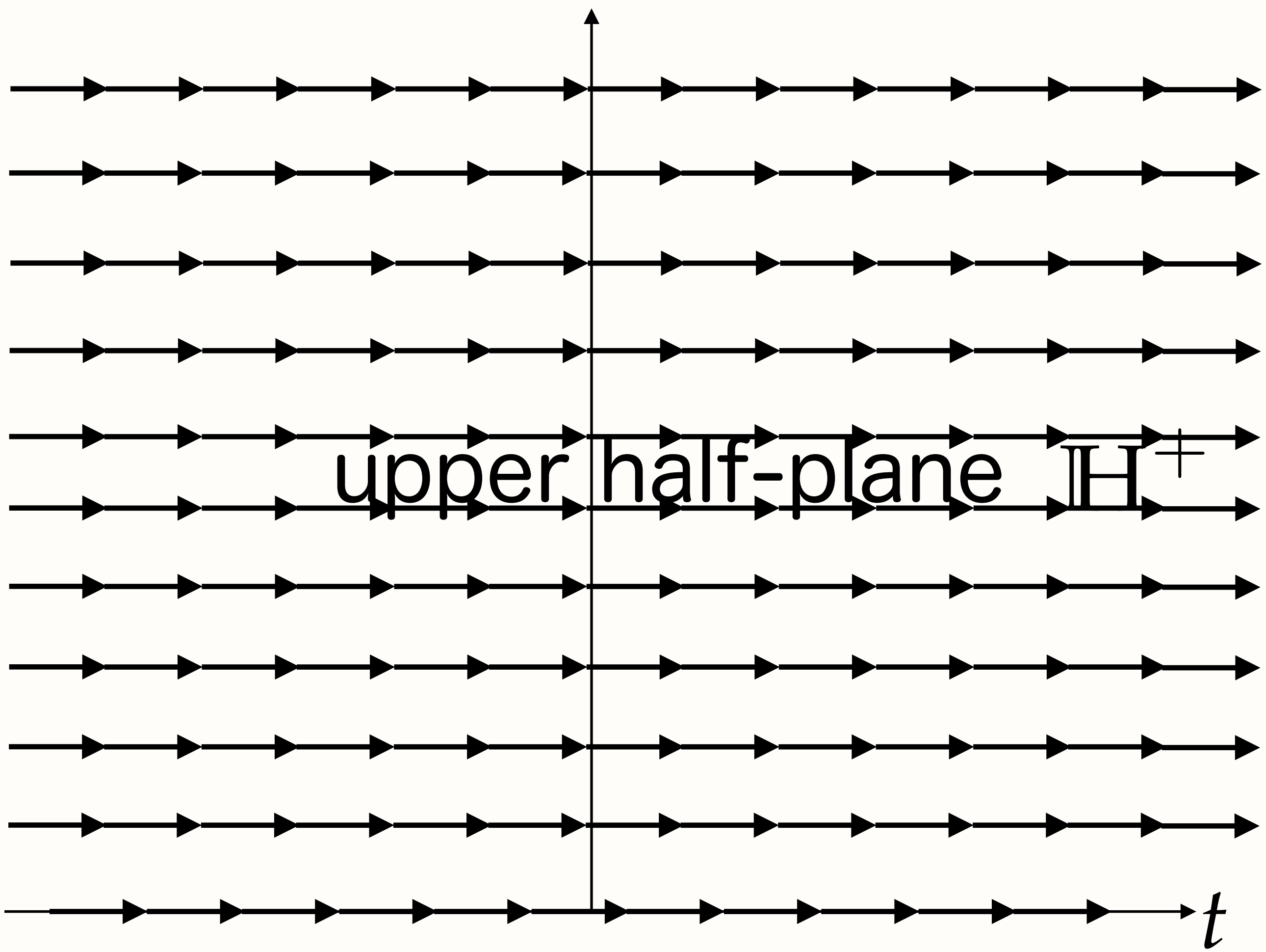
Outlook

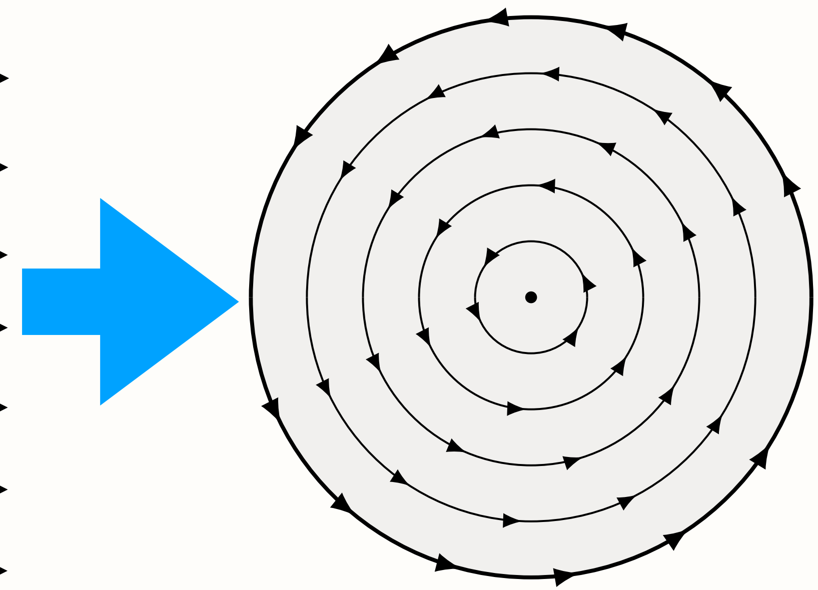
Time foliation over curved space

Two or more “time”s

Continuous Virasoro algebra

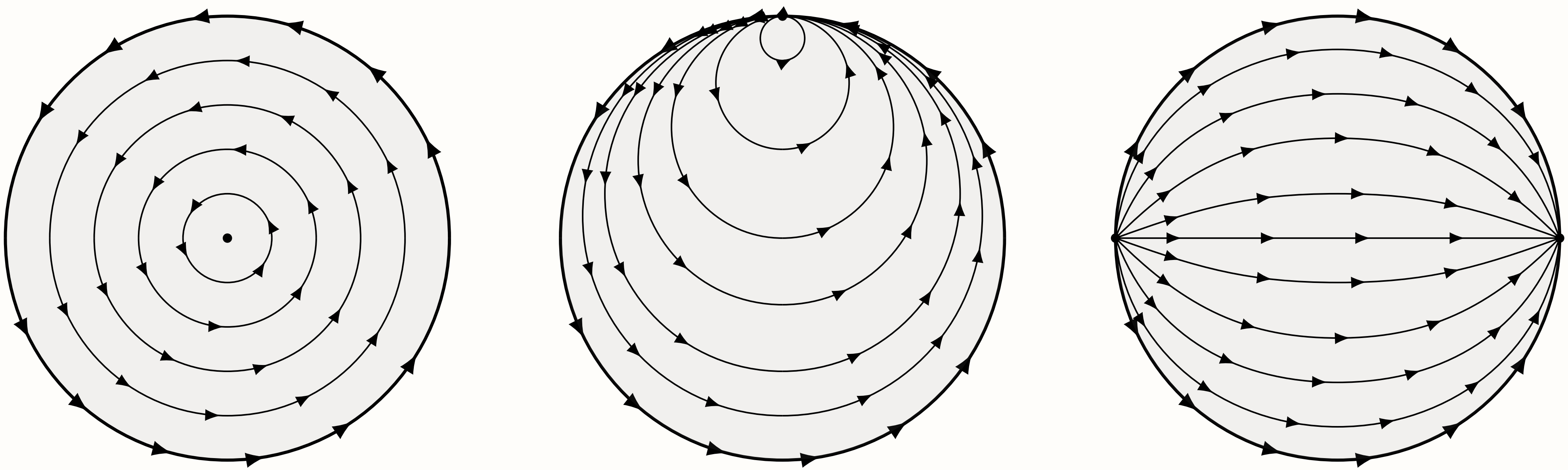
Hamiltonian Action



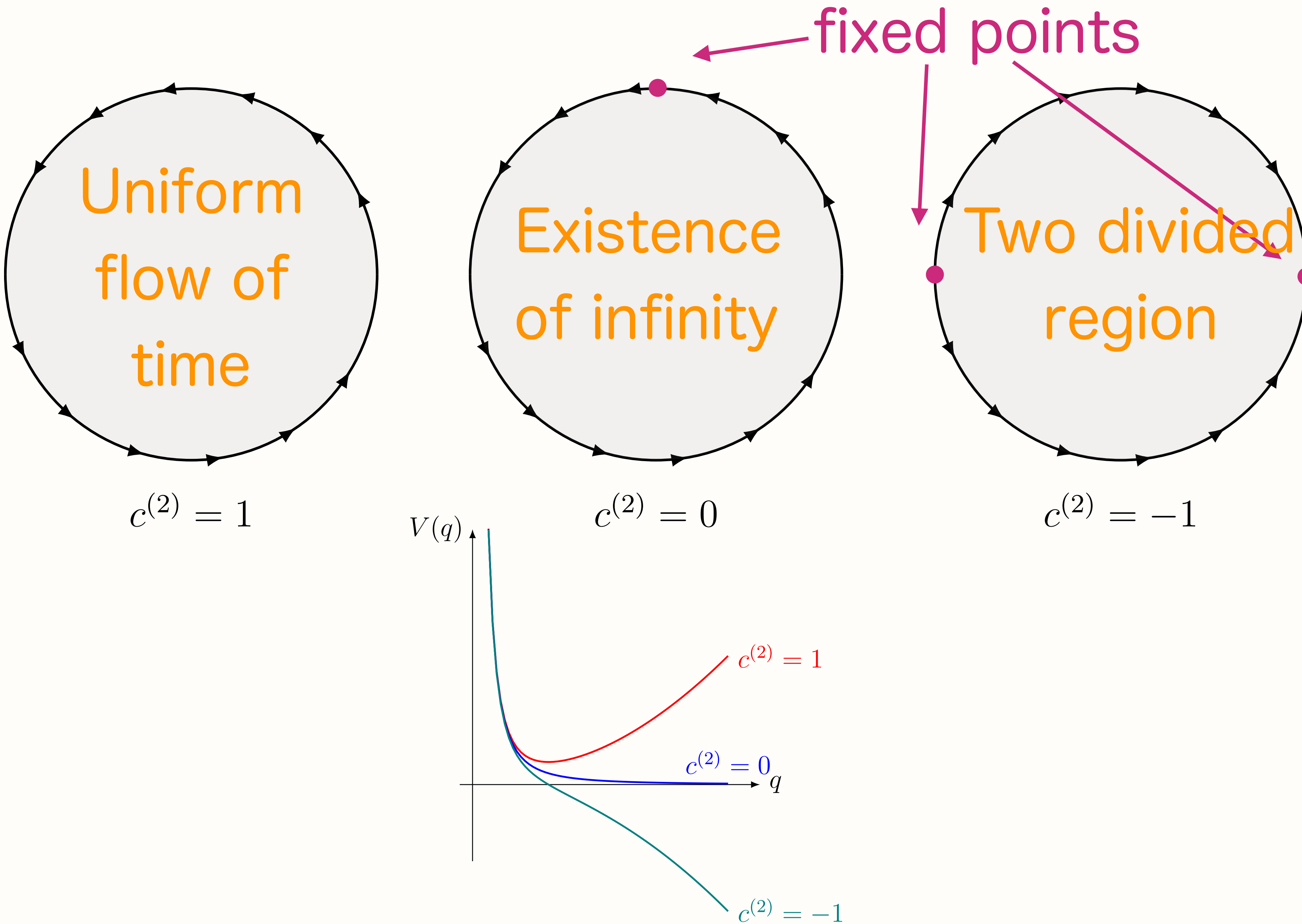


Map to
Poincare Disk

Automorphism on Poincare Disk



$c^{(2)} = 1$ $\qquad$ $c^{(2)} = 0$ $\qquad$ $c^{(2)} = -1$


fixed points
Uniform flow of time
Existence of infinity
Two divided region
$c^{(2)} = 1$
$c^{(2)} = 0$
$c^{(2)} = -1$
$V(q)$
q
$c^{(2)} = 1$
$c^{(2)} = 0$
$c^{(2)} = -1$